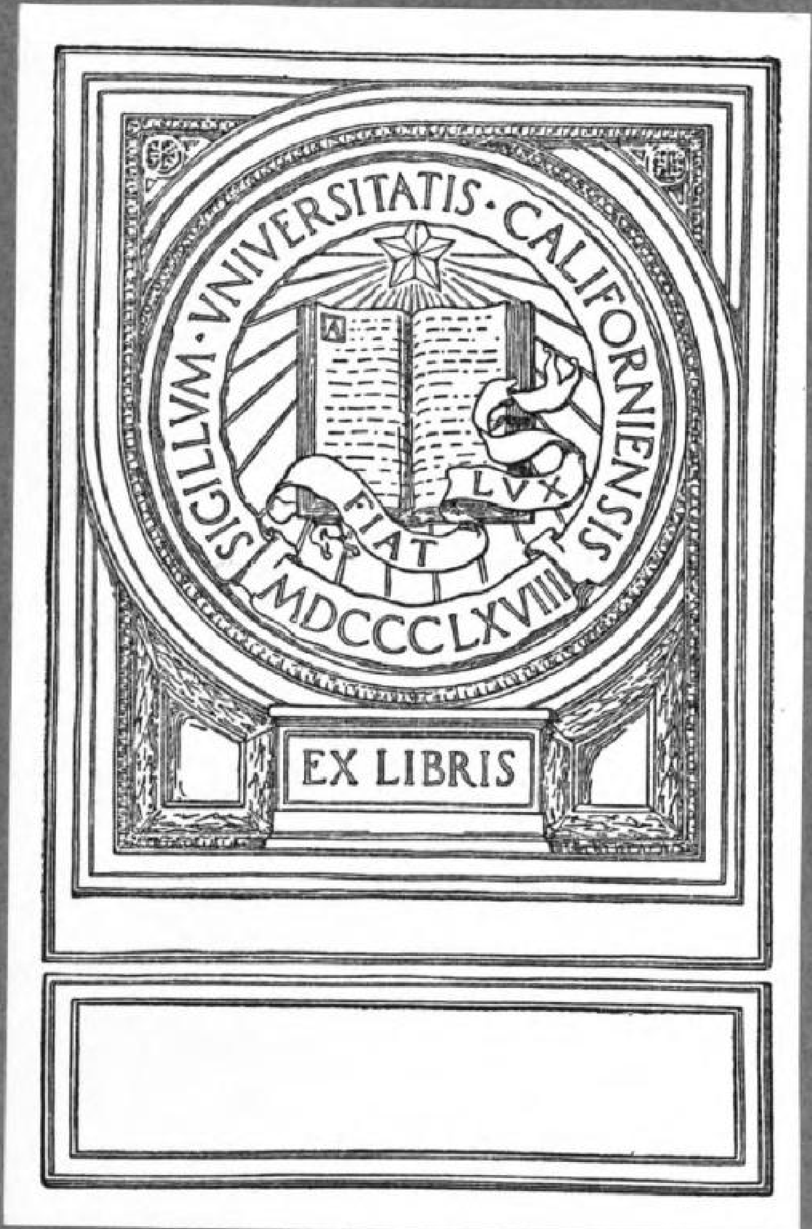
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THE MATHEMATICS OF FINANCE

BY

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EDITOR'S INTRODUCTION

Courses in the Mathematics of Finance (or Investment) have of recent years become a recognized unit of collegiate instruction in mathematics. A considerable number of well-organized textbooks designed for such courses are already on the market. There is a very definite feeling among the teachers of this subject, however, that all hitherto existing texts suffer from a serious fault: They are all burdened with a mass of formulas and rules which not only impose a severe strain on the memory of the student, but (and this is pedagogically far worse) they lead almost inevitably to mere mechanical substitution and to a lack of understanding of the fundamental principles and a consequent absence of any real power of analysis. Some teachers have perhaps felt that such a mass of formulas is inevitable, that it is inherent in the enormous variety of applications of simple and compound interest and discount.

The present text proves, however, that such is not the case. One of its most gratifying features lies in the fact that the authors have succeeded in showing how all the varied standard types of problems that arise in the mathematics of finance can be solved by the application of a small number of fundamental formulas and their simple transformations. This is a gain that will, I feel confident, profoundly affect the future teaching of this subject. It places the emphasis of the instruction where it should be, on a thorough understanding of principles. The latter once mastered, the rest is comparatively easy; without such mastery, however, any apparent success attained in the teaching of this subject is sure to be to a large extent illusory.

The feature above referred to will recommend the present text to every conscientious teacher who has the real welfare of his students at heart. He will find, on examining the text further, that the authors have succeeded in so organizing their material that, after mastering certain fundamental principles in the first chapter, the

student is led gradually from simple applications to the more complete — another obviously desirable feature.

Finally, he will find a wealth of carefully graded problems that are real, taken from actual business transactions.

J. W. Young

AUTHORS' PREFACE

The outstanding features of this book are the following:

(*a*) General formulas are derived for the values of single sums of money, annuities certain, life annuities, and life insurances.

(*b*) Line diagrams are extensively used as an aid to the analysis of problems.

(*c*) Solutions of many typical examples are given in the text.

(*d*) The best method of computation is shown in the solution of examples.

(*e*) The exercises are chosen to illustrate practical problems of business.

(*f*) An excellent collection of tables is given, and

(*g*) The articles are so arranged as to make lesson assignments convenient, and an abridged course easily planned.

The authors have found, from their experience in teaching the elements of the mathematics of finance, that students learn to derive and to use general formulas for the values of annuities certain, life annuities, and life insurances as easily as they learn to derive and to use special formulas. The verbal statements which are given for most of these formulas are helpful to some students both in memorizing and in using them. Since the number of these general formulas is small, students are able to solve all problems in finance by the use of only a few principles. A good illustration of the advantage of the general formula is afforded by formula (1), Chapter IV, for the value of any life annuity based on a single life. By means of this formula, the values of ordinary, deferred, and forborne life annuities immediate and life annuities due can be written at once. The use of a small number of general principles permits also a greater amount of attention to the practical applications than is usually given in an elementary text.

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are also indebted to Mr. J. C. Rietz, actuary of the Midland Mutual Life Insurance Company, and to Mr. H. C. Fetsch, actuary of the Ohio State Life Insurance Company, for their assistance in the preparation of the manuscript, Chapters IV and V. Prof. J. W. Glover of the University of Michigan, and his publisher, George Wahr, have kindly permitted the authors to include portions of his *Tables of Applied Mathematics in Finance, Insurance and Statistics*, a book which is invaluable to a student of these subjects.

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MATHEMATICS OF FINANCE

CHAPTER I

INTEREST AND DISCOUNT

1. The problem of finding the value of a sum of money. If a principal of \$100 is invested for one year at 6% simple interest, the amount of the investment at the end of the year is \$106. If \$106 is the amount of an investment for one year at 6% simple interest, the value of the investment at the beginning of the year is \$100. These simple examples from arithmetic illustrate a problem whose solution is of primary importance in the mathematics of finance. If the value of a sum of money is known at a given time, this problem is to find its value at any other time. If n denotes the number of years, called the *term in years*, between the given time and the time at which the value is to be found, and if S denotes the larger and P the smaller of the values at the two times, then the problem may be stated in the form: to find S when P and n are given and to find P when S and n are given. S is called the *amount* or the *accumulated value* of P for n years and P is called the *present value* or the *discounted value* of S for n years. The operation by which S is found is called *accumulating;* that by which P is found is called *discounting.* The accumulating operation increases the value of a sum of money, the discounting operation decreases its value. These operations are based on the fundamental assumption that money is constantly productive.

The value found by accumulating or discounting a given sum of money depends on the rate as well as on the term n. The rate may be either an interest rate or a discount rate. In this chapter four methods of accumulating and four methods of discounting are presented. These methods are based on the principles of simple

interest, simple discount, compound interest, and compound discount. The primary importance of these operations lies in the fact that by means of them the equations needed to find the unknowns in various types of problems in the elements of finance can be readily determined. A thorough mastery of them is essential. Before presenting them it is necessary to define interest and discount; and interest and discount rates. This is done in Art. 2.

2. Interest and discount; interest and discount rates. In the numerical examples in Art. 1, P is \$100, S is \$106, n is 1, and $S - P$ is \$6; this difference, \$6, is called the interest on \$100 for one year and also the discount on \$106 for one year. In general $S - P$ is called the *interest on P* for n years and the *discount on S* for n years. When $S - P$ represents the interest on P, it will be denoted by I; when it represents the discount on S, it will be denoted by D. That is,

$$\text{Interest on } P = I = S - P = D = \text{discount on } S.$$

Interest and discount rates are defined with respect to some definite period of time. In practice the period chosen is a year or a part of a year. The interest rate per period is the number by which any sum must be multiplied to find the interest on it for the period; the discount rate per period is the number by which any sum must be multiplied to find the discount on it for the period. If \$1 is taken for the sum, these definitions become:

The interest rate per period equals in numerical value the interest on \$1 *for the period; the discount rate per period equals in numerical value the discount on* \$1 *for the period.*

In the above examples the interest rate per year is $\frac{6}{100}$ and the discount rate per year is $\frac{6}{106}$.

An interest or a discount rate may be expressed as a per cent, as a decimal fraction, or as a common fraction. For example, $6\% = .06 = \frac{6}{100}$. In computations the decimal fraction form is usually the more convenient; in some cases, however, the common fraction form may be used to advantage. In verbal statements the per cent form is generally employed.

EXERCISES

1. The interest on \$100 for one year is \$4.25. Find the interest rate as a per cent, as a decimal fraction, and as a common fraction.

Ans. $4\frac{1}{4}\%$; .0425; $\frac{17}{400}$.

2. The discount on \$130 for one year is \$5.20. Find the discount rate as a per cent, as a decimal fraction, as a common fraction. Ans. 4%; .04; $\frac{4}{100}$.

3. If \$325 at simple interest amounts in one year to \$347.75, express the interest rate in each of the three ways. Ans. 7%; .07; $\frac{7}{100}$.

4. If $P = \$125$, $S = \$132.50$, and $n = 1$, find the interest and discount rates as common fractions. Ans. $\frac{3}{50}$; $\frac{3}{53}$.

5. The present value of \$125, due in one year, is \$117.98. Find the interest and discount rates each to four decimals.[1] Ans. .0595; .0562.

3. Simple interest formulas. In defining a simple interest rate, one year is the period chosen. If i denotes the simple interest rate per year, then, by Art. 2, Pi is the simple interest on P for one year. In general, if I represents the simple interest on P for n years, the value of I is defined by

$S - P = I$

$$I = Pni \tag{1}$$

This equation shows that I varies as P, n, and i.

By Art. 2, $S = P + I$. Replacing I by Pni gives

$$S = P(1 + ni) \tag{2}$$

or, solving for P,

$$P = \frac{S}{(1 + ni)} \tag{2'}$$

Formulas (2) and (2′) may be stated verbally in the form: *To find the amount of any sum for n years at the simple interest rate i, multiply this sum by* $1 + ni$*; to find the present value of any sum due in n years at the simple interest rate i, divide this sum by* $1 + ni$.

EXERCISES

1. Find the simple interest on \$125.65 at $4\frac{1}{2}\%$ for 1 year, for $\frac{1}{2}$ year, for $\frac{1}{4}$ year. What is the amount in each case?

2. Find the simple interest on \$250 at $5\frac{1}{2}\%$ for 9 months. What is the amount? Ans. \$10.31; \$260.31.

[1] The phrase "to four decimals" should be interpreted to mean to the nearest digit in the fourth decimal place.

3. If \$325.50 amounted to \$345.03 in 1 year at simple interest, find the interest rate used. Ans. 6%.

4. How long will it take \$100 to accumulate to \$125 at 5% simple interest? How long to accumulate to \$200? Ans. 5 years; 20 years.

5. What principal invested at $5\frac{3}{4}$% will amount to \$1235.50 in 1 year and 9 months? Ans. \$1122.54.

6. Find the present value of a non-interest-bearing note for \$1100 due in 9 months, if a simple interest rate of 6% is used. Ans. \$1052.63.

7. Same as Exercise 6 except that the note bears simple interest at 5%. Ans. \$1092.11.

8. How long will it take any sum P to accumulate to $2P$ at a simple interest rate i?

4. Simple discount formulas. In defining a simple discount rate, one year is the period usually chosen. If d denotes the simple discount rate per year, then, by Art. 2, Sd is the simple discount on S for one year. In general, if D represents the simple discount on S for n years, the value of D is defined by

$$D = Snd \tag{3}$$

This equation shows that D varies as S, n, and d.

By Art. 2, $P = S - D$. Replacing D by Snd gives

$$P = S(1 - nd) \tag{4}$$

or, solving for S,

$$S = \frac{P}{(1 - nd)} \tag{4'}$$

Formulas (4) and (4′) may be stated verbally in the form: *To find the present value of any sum due in n years at the simple discount rate d, multiply this sum by* $(1 - nd)$; *to find the amount of any sum for n years at the simple discount rate d, divide this sum by* $(1 - nd)$.

EXERCISES

1. Find the simple discount on \$150.60 at $4\frac{1}{2}$% for 1 year, for $\frac{1}{2}$ year, for $\frac{1}{4}$ year. What is the present value in each case?

2. Find the simple discount on \$250 at $5\frac{1}{2}$% for 9 months. What is the present value? Ans. \$10.31; \$239.69.

3. If the present value of \$125 due in 1 year is \$118.75, find the discount rate used. Ans. 5%.

4. The present value of \$200 is \$192.50. If the rate of discount is 5%, find the time in months for which the discount is calculated. Ans. 9 months.

5. What sum discounted at $5\frac{3}{4}\%$ for 1 year and 9 months has a present value of \$1235.50? Ans. \$1373.73.

6. If a wholesaler gives a discount of 15% and 10% off the list price, find the equivalent simple discount rate. Ans. 23.5%.

7. Same as Exercise 6, except that an additional discount of 2% for cash is allowed.

8. How long will it take any sum S to become $\frac{S}{2}$ at a simple discount rate d? How long to become zero?

5. Computation of simple interest and simple discount, exact and ordinary. In formulas (1) and (3) for computing simple interest and simple discount, n represents the number of years in the term. In business transactions n is usually equal to or less than unity. When the number of months in the term is given, n is found by dividing this number by 12. When the number of days in the term is given, n is found by dividing this number by the number of days in a year. In some transactions a year is regarded as made up of twelve months of thirty days each or 360 days; in others, of twelve calendar months or 365 days. These two ways of regarding a year lead to two values for n for any definite number of days b, namely, $\frac{b}{360}$ and $\frac{b}{365}$. When n in formulas (1) and (3) is replaced by $\frac{b}{365}$, they become

$$I = \frac{Pbi}{365}, \qquad D = \frac{Sbd}{365}$$

and the values of I and D so determined are called *exact simple interest* and *exact simple discount* for a term of b days. When n is replaced by $\frac{b}{360}$ the formulas become

$$I = \frac{Pbi}{360}, \qquad D = \frac{Sbd}{360}$$

and the values of I and D so determined are called *ordinary simple interest* and *ordinary simple discount* for a term of b days.

Excellent tables giving the ordinary and exact interest and discount when $P = S = 1$ have been constructed for values of b ranging from 1 to 360 or 365 and for various values of i and d. Such tables greatly facilitate the computation of simple interest and simple discount.

There is variation in practice, even among banks, as to when the exact or the ordinary formulas are used in determining interest or discount.

In short-term loans made by banks ordinary simple discount is frequently used. In such cases the borrower executes a note for a stated amount which does not bear interest and the bank immediately discounts this note at a quoted rate. For example, if the rate quoted is 7% and the note states that the borrower is to pay $100 to the bank at the end of one year, the bank gives the borrower $100 - 7 = \$93$. The amount deducted from the face of the note in such cases, here $7, is sometimes spoken of as interest in advance; it is in reality, however, the discount on the face of the note.

Exact simple discount is used by the Federal Reserve banks in rediscounting notes presented by those banks which are members of the Federal Reserve System.

Exact simple interest is customarily used by banks which pay interest on daily balances.

Ordinary simple interest is used by banks on money loaned by them on demand notes.

When the term during which interest or discount is to be computed is stated in days, b has a unique value in the above formulas. For this reason banks are more and more adopting the practice of stating the term in days. When the term is the interval between two given dates, the value of b depends usually on whether calendar months or months of thirty days are used in its determination. When calendar months are used, b is the exact number of days between the two dates. This exact number may be found by adding the number of days in each month between the given dates or by use of Table II, which gives the number of each day of the year. When thirty-day months are used, the number of days can be found by the method presented in arithmetic. By Table II the exact

number of days from November 22, 1923, to February 10, 1924, is $39 + 41 = 80$ days. When thirty-day months are used, the number of days between these two dates is found as follows:

YEAR	MONTH	DAY	
1924	2	10	
1923	11	22	
	2	18	$= 60 + 18 = 78$ days.

In general if b_1 denotes the number of days in any term when calendar months are used and b_2 the number when thirty-day months are used, then b_1 may be greater than, equal to, or less than b_2. There are four possible values for n when b_1 and b_2 are different, namely, $\frac{b_1}{365}, \frac{b_2}{360}, \frac{b_1}{360}, \frac{b_2}{365}$. In the first of these, calendar months are used in determining both the numerator and the denominator; in the second, thirty-day months are used in determining both. The last two do not possess this uniformity. Financial institutions ordinarily use $\frac{b_1}{360}$ when calculating interest on accounts receivable and $\frac{b_1}{365}$ on accounts payable. In transactions not involving financial institutions, such as those between individuals, $\frac{b_2}{360}$ is frequently used.

EXERCISES *

1. Find the number of days from November 22, 1923, to March 10, 1924, by calendar months and by thirty-day months. Ans. 109; 108.

2. A note dated February 5, 1923 is due in 90 days. Find its date of maturity. If it is due in 3 months, find its date of maturity.

Ans. May 6; May 5.

3. A note dated March 30 is paid September 5. Find the time in days for which the interest on this note is calculated, first, if 30-day months are used, and second, if the exact number of days are used. Ans. 155; 159.

4. In some states a note which falls due on Sunday or a holiday is legally payable on the next business day, the interest or discount being computed to the day of payment. In one of these states a note due in 30 days is given to

* In these exercises use the ordinary simple interest and discount formulas and the exact number of days between dates unless otherwise specified.

a bank on August 4. If September 3 falls on Sunday, for how many days will the discount be calculated; for how many days if September 4 is Labor Day? Ans. 31; 32.

5. A note, dated January 3, 1920, fell due in 90 days. What was the date of maturity? It was discounted by a bank on February 1. For how many days was the discount calculated? Ans. April 2; 61.

6. A note was given to a bank on March 30 for 90 days. On April 5 it was rediscounted by a Federal Reserve bank. Find the number of days for which each discount was calculated. When was the note due? Ans. 90; 84; June 28.

7. Find the ordinary simple interest on a note, dated January 12, 1923, for \$1250.75 at 6% for 108 days. When was it due? Ans. \$22.51; April 30, 1923.

8. Find the numbers to fill the blanks, using the ordinary simple interest formula:

P	i	Term	S
1250		65 days	1265
1250	.0425		1350
1000	.045	25 days	
	.04375	60 days	1250
750.50		195 days	785.25
65.35	.05		71.55
75.25	.05	151 days	
	.035	12 days	65.15

9. On August 3, 1924, A gave B a note for \$500. It was paid March 2, 1925. Find the ordinary simple interest due if 30-day months and a 6% interest rate are used. Ans. \$17.42.

10. A note for \$100 due in 30 days bore interest at 8% from date of maturity. The note was paid 58 days after its date of issue. Find the ordinary simple interest due. Ans. \$.62.

11. A note for \$100 due in 30 days is given to a bank which at once discounts it at 7%. How much does the maker of the note receive for it? Ans. \$99.42.

12. A note dated January 31, 1923, and due in 30 days is discounted by a bank on February 15. If the face of the note is \$150 and it bears 6% interest from date, how much does the holder of the note receive for it if the bank's rate of discount is 7%? Ans. \$150.31.

13. A non-interest-bearing note for $100 was sold for $98.95, 50 days before it was due. Find the rate of discount correct to four decimals. Ans. .0756.

14. A note for $1275 was dated July 3. It bore interest at 6% from date and was due in 90 days. It was discounted August 31. If the holder of the note received $1282.35, find the rate of discount correct to four decimals. Ans. .1057.

15. Find the face of a note which was due in 30 days, bore 6% interest, and which when discounted by a bank on the day it was made at 7% yielded the holder $126.76. Ans. $126.87.

16. The holder of a non-interest-bearing note dated January 15, 1921, and due in 6 months discounted it at a bank on March 1 at 7%. The bank's discount on the note was $85.36. What was the face of the note? Ans. $3227.90.

17. Solve Exercise 16 on the hypothesis that the note bore 6% simple interest from its date of issue. Ans. $3133.88.

18. A note for $125 due in 50 days bore simple interest at 7% from date of issue. It was discounted 35 days before its date of maturity. Find the bank's rate of discount, correct to four decimals, if the holder of the note received $125.30. Ans. .0750.

19. A note for $450, which bore simple interest from date and which was due in 30 days, was discounted 20 days before maturity at $7\frac{1}{2}$%. If the proceeds of the note was $450.37, find to four decimals the note's rate of interest. Ans. .0600.

20. A certain bank pays 2% exact simple interest on daily balances of even hundreds when they amount to $1000 or more. On a day that a man's balance was $1250.67 find his interest credit.

21. A bank book showed the following balances for the first six days of February:

Feb. 1,	$ 875.35	Feb. 4,	$2635.28
Feb. 2,	1872.38	Feb. 5,	3625.24
Feb. 3,	1555.80	Feb. 6,	2732.86

If the bank pays $1\frac{1}{2}$% exact simple interest on daily balances on even hundreds when they are $500 or more, find the interest credit for each day.

22. Find the numbers to fill the blanks:

P	i	Term	Ordinary Simple Interest	Exact Simple Interest
1256.35	.05	100 days		
675.27	.055	60 days		
2367.85	.0475	108 days		

23. A Federal Reserve bank bought a \$1000 non-interest-bearing note 20 days before it was due. Find the purchase price of the note if exact simple discount at $4\frac{1}{2}\%$ was used. Ans. \$997.67.

24. A note for \$1000 due in 60 days from date of issue with interest at 5% was bought by a Federal Reserve bank 25 days before its maturity. Find the purchase price of the note if exact simple discount at $4\frac{1}{2}\%$ was used.

Ans. \$1005.39.

25. A non-interest-bearing note for \$12,000 due in 30 days was purchased by a bank on its day of issue. It was rediscounted by a Federal Reserve bank 20 days before it was due. Find each purchase price if ordinary simple discount at 7% was used in the first transaction and exact simple discount at $4\frac{1}{2}\%$ in the second.

26. A note dated January 10, 1922, for \$750 bore 7% interest from date and was due in 90 days. It was discounted by a bank at $7\frac{1}{2}\%$ on February 1 and by a Federal Reserve bank on February 15 at $5\frac{1}{2}\%$. How much did the bank pay for the note and how much did it receive from the Reserve bank, if ordinary simple discount was used in the first transaction and exact simple discount in the second? Ans. \$752.32; \$756.83.

27. A note for \$175, due in 30 days, bore 5% interest. It was discounted the day it was made at 7% by a bank which rediscounted it the same day at $5\frac{1}{2}\%$ with a Federal Reserve bank. Find the ordinary simple discount involved in the first transaction and the exact simple discount in the second.

Ans. \$1.03; \$.76.

28. A note for \$1000 bearing interest at 7% fell due in 90 days. It was discounted by a bank at $5\frac{1}{2}\%$ on the day it was made, and rediscounted on the same day by a Federal Reserve bank at the same rate. How much did the bank gain by the transaction if it uses ordinary simple discount and the Federal Reserve bank uses exact simple discount? Ans. \$.19.

29. If I_o represents the ordinary simple interest on P at rate i for a term n and I_e represents the exact simple interest for the same P, i, and n, show that

$$I_o = \tfrac{73}{72} I_e$$
$$I_e = \tfrac{72}{73} I_o$$

30. Use the formulas in Exercise 29 to check the results in Exercise 22.

31. Derive formulas analogous to those in Exercise 29 for ordinary simple discount and exact simple discount.

6. Problems based on the simple interest and the simple discount formulas. The simple interest formulas

$$I = Pni \tag{1}$$

$$S = P(1 + ni) \tag{2}$$

contain five letters, and when the values of three are known, the values of the other two can be found by solving equations (1) and (2) simultaneously for them. The problem of finding I and S when P, n, and i are given is the one of most frequent occurrence among problems of this type. When these formulas are used in business transactions, the term n is usually one year or less.

The simple discount formulas

$$D = Snd \tag{3}$$

$$P = S(1 - nd) \tag{4}$$

lead to the solution of analogous problems in discount. Here also the term is usually one year or less in business transactions. An important exception is found in the straight line or simple discount method of computing depreciation (Art. 62, Chapter III).

Additional problems can be solved by using any two or more of the equations (1), (2), (3), and (4). An important one of this type relates to corresponding rates. *The simple interest rate i and the simple discount rate d are said to be corresponding for a given value of n when each leads to the same accumulated value, S, of P in n years.* Equating the values of S given by formulas (2) and (4), and dividing by P, gives the following equation connecting the corresponding rates i and d:

$$1 + ni = \frac{1}{1 - nd} \tag{5}$$

or, solving for d,

$$d = \frac{i}{1 + ni} \tag{5'}$$

or, solving for i,

$$i = \frac{d}{1 - nd} \tag{5''}$$

Equations (5′) and (5″) show that, when i and d are corresponding rates, d is the present value of i for n years at the simple interest rate i, and i is the amount of d for n years at the simple discount rate d. In particular, when $n = 1$, $d = \frac{6}{106}$ corresponds to $i = \frac{6}{100}$ and $i = \frac{6}{94}$ corresponds to $d = \frac{6}{100}$. This is as would be expected, as the borrower of a dollar may pay for the use of it d dollars at the beginning of the term or i dollars at the end of the term. Clearly, then, d is the discounted value of i and i is the accumulated value of d.

EXERCISES

1. Find the present value of \$500 due in 9 months without interest if the discount rate is 5%; if the interest rate is 5%; if the discount rate is $5\frac{1}{2}$%; if the interest rate is $5\frac{1}{2}$%.

2. Find the accumulated value of \$125 for 1 year and 3 months, if the interest rate is 6%; find the accumulated value, if the discount rate is 6%.

3. Show that if i and d are corresponding and $n = 1$,

$$(a)\ i = \frac{d}{1-d} = d + d^2 + d^3 + \cdots$$

$$(b)\ d = \frac{i}{1+i} = i - i^2 + i^3 - \cdots$$

4. If d is $4\frac{1}{2}$% and $P = \$100$, find S if $n = 1$; find the corresponding i.

5. Find the numbers to fill the blanks, i and d being corresponding rates:

d	i	n (years)
.04		1
.04		$\frac{1}{2}$
	.06	$\frac{1}{2}$
	.07	1
.05		1

6. Find the numbers to fill the blanks, i and d being corresponding rates:

P	i	S	d	n (years)
100		103		$\frac{1}{2}$
	.035	112		1
125	.045			$\frac{1}{4}$
		118	.035	$\frac{3}{4}$
135	.06	140		
	.05	125	.0475	
110	.05		.0475	
120		126	.045	
175			.055	1

7. If a bank discounts a 30-day note at 7%, find the corresponding simple interest rate earned by the bank. Use a year of 360 days.

8. Same as Exercise 7 except that the note is a 60-day note; a 90-day note.

9. The interest on \$56.75 for one year is \$4.25; find the corresponding interest and discount rates correct to four decimals. Ans. .0697; .0749.

10. The discount on \$56.75 for 6 months is \$2.10; find the corresponding simple interest and discount rates correct to four decimals.

Ans. .0769; .0740.

11. A mortgage note for \$1000 due in 1 year is executed in favor of a second mortgage loan company. The company discounts the face of the note at 7% and in addition charges a commission of 10% on the face of the note for making the loan. Find the simple interest rate earned by the company.

12. Solve $S = P(1 + ni)$ for each of the four letters.

13. Solve $P = S(1 - nd)$ for each of the four letters.

14. Show that the following relations hold where i and d are corresponding rates for a term of n years:

$$(a)\quad Pi = Sd$$

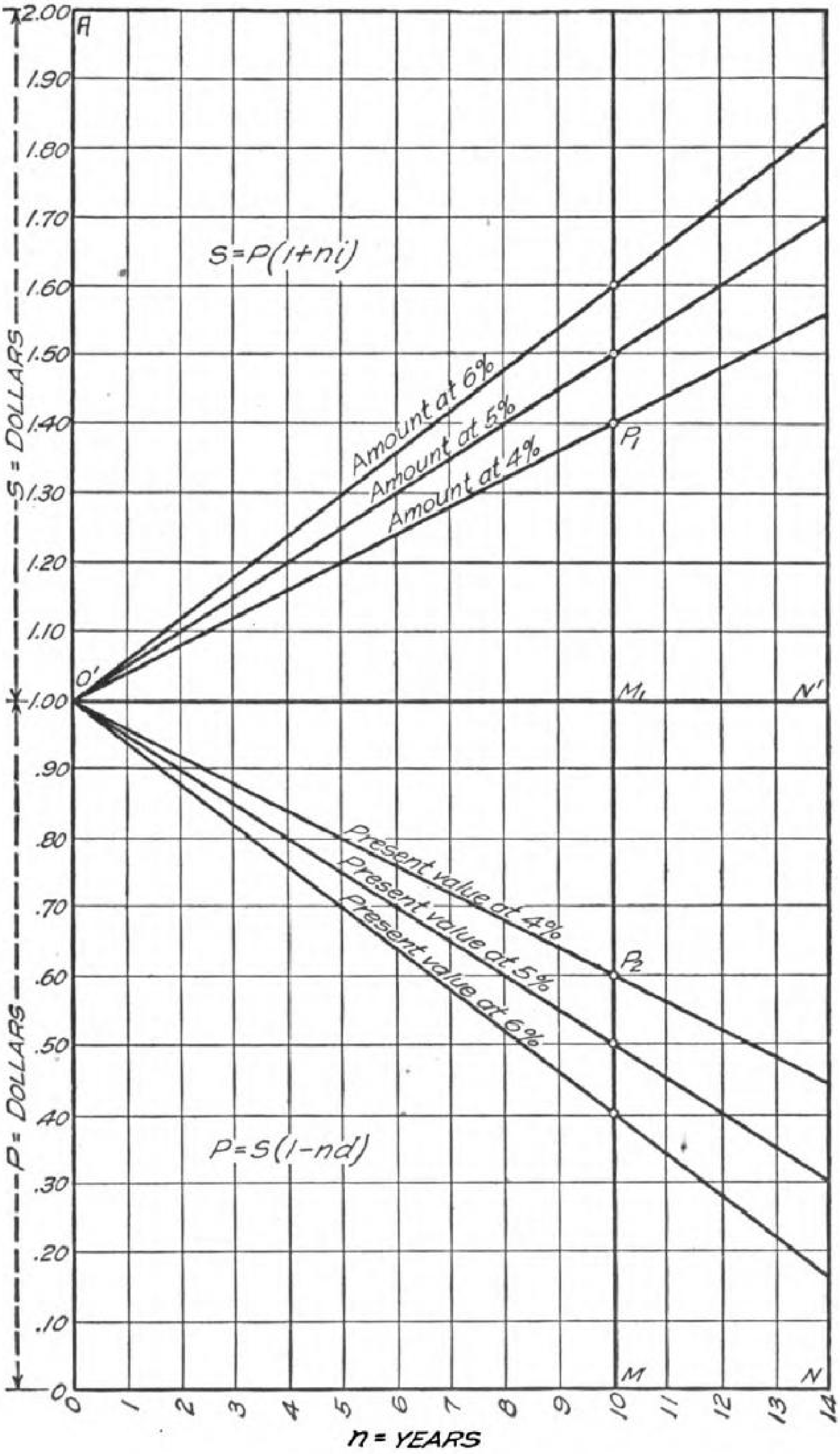
$$(b)\quad S = \frac{P}{1 - nd} = P(1 + ni)$$

$$(c)\quad P = \frac{S}{1 + ni} = S(1 - nd)$$

State verbally the properties of corresponding rates expressed by relations (*a*) and (*c*).

15. The accumulated value of P for n years may be found by using the interest rate or the discount rate; the present value of S, by using either the interest rate or the discount rate. Verify these statements by the use of (*b*) and (*c*) of Exercise 14.

7. Graphs of the simple interest and the simple discount formulas. When P and i are given, equations (1) and (2) are of the first degree and hence have straight lines for their graphs. Similarly, when S and d are given, equations (3) and (4) have straight-line graphs. Figure 1 shows the graphs of $S = P(1 + ni)$ for $P = 1$, $i = .04$, .05, and .06, and of $P = S(1 - nd)$ for $S = 1$, $d = .04$, .05, and .06 with respect to the axes ON and OA. The values of I and D are represented by the distances from $O'N'$. For example, MP_1 represents the amount and M_1P_1 the simple interest on \$1 for 10 years at 4% simple interest, and MP_2 represents the present value and P_2M_1 the simple discount on \$1 for 10 years at 4% simple discount.



S = DOLLARS
P = DOLLARS
S=P(1+ni)
P=S(1-nd)
Amount at 6%
Amount at 5%
Amount at 4%
Present value at 4%
Present value at 5%
Present value at 6%
A
O'
M₁
N'
P₁
P₂
M
N
O
n = YEARS

FIGURE 1

It may be noted that the graphs of $S = P(1 + ni)$ when S and i are given and of $P = S(1 - nd)$ when P and d are given are not straight lines, since the equations are of the second degree in the other letters. In these cases the graphs are hyperbolas.

EXERCISES

1. Construct the graph of $S = P(1 + ni)$ if $P = 200$ and $i = 4\frac{1}{2}\%$.
2. Construct the graph of $P = S(1 - nd)$ if $S = 1000$ and $d = 10\%$.
3. Construct the graph of $S = P(1 + ni)$ if $S = 100$ and $i = 5\%$.

8. Compound interest; compound discount. In computing simple interest or simple discount, the principal on which interest or discount is computed is constant during the term. In computing compound interest, the principal on which interest is computed is increased by the amount of the interest payment each time the interest due is paid during the term. The *compound interest*, $S - P$, for the term n on an original principal P, is the total amount of interest computed in this way. Similarly, in computing compound discount, the principal on which discount is computed is decreased by the amount of the discount payment each time the discount due is paid during the term, and the *compound discount*, $S - P$, for the term n on S, is the total amount of discount computed in this way.

In compound interest or discount, the interest or discount payments are converted into principal annually, semi-annually, quarterly, or at some other periodic interval, the interval between successive conversions being called a *conversion* period. The number of conversion periods per year will be denoted by m, and the interest rate and the discount rate per conversion period will be denoted by $\frac{j}{m}$ and $\frac{f}{m}$ respectively. It follows that j is m times the interest rate per conversion period and f is m times the discount rate per period. The number, j, is called the *nominal interest rate per year*, and the number, f, the *nominal discount rate per year*, since in business transactions it is customary to quote or name these rates with the number of conversions rather than the rates per conversion period. When $m = 1$, the nominal interest rate,

j, is called the *effective interest rate* and is denoted by i. Likewise when $m = 1$, the nominal discount rate, f, is called the *effective discount rate* and is denoted by d. For example, a nominal interest rate of 6% converted quarterly means an interest rate of $1\frac{1}{2}\%$ per quarter year, while an effective interest rate of 6% means an interest rate of 6% per year. A nominal interest rate, j, converted m times per year is sometimes denoted by the symbol, $j_{(m)}$, and a nominal discount rate, f, converted m times per year, by $f_{(m)}$. Numerical rates may be denoted more conveniently by enclosing them within parentheses. For example, ($j = .06$, $m = 4$) denotes a nominal interest rate of 6% converted four times per year.

It should be noted that the nominal rates per year are not the rates per year as defined in Art. 2 except when m is unity. For example, at ($j = .06$, $m = 2$), \$1 amounts to \$1.03 in a half year and this amounts to $1.03 + 1.03 \cdot (.03) =$ \$1.0609 in another half year, so that the total interest earned on \$1 in one year is \$.0609 and the rate per year is 6.09%. Two compound interest rates with different conversion periods which lead to the same amount of P in one year are said to be corresponding. In the example given 6% converted semi-annually corresponds to 6.09% converted annually; in other words, the nominal rate ($j = .06$, $m = 2$) corresponds to the effective rate $i = .0609$. A general definition of corresponding rates and a method for determining them will be given in Art. 13.

If an investment is of such a nature that the interest or discount payments cannot be used to change the principal as they are in compound interest or compound discount, these payments may be put into other investments. When they are put into investments having the same conversion periods and the same rate per period as the original investment, the total amount of interest or discount realized is the same as that given by compound interest or compound discount. When, however, the payments of interest or discount are put into investments bearing rates differing from that of the original investment, the results are not the same in general. The compound interest and discount formulas are developed in the next two articles. The more general case involving different interest rates will be considered in Art. 31, Chapter II.

9. Compound interest formulas. By Art. 8, 1 at the beginning of any interest conversion period amounts to $1 + \frac{j}{m}$ at the end of the period, and hence A at the beginning amounts to $A\left(1 + \frac{j}{m}\right)$ at the end. It follows that

P amounts to $P\left(1 + \frac{j}{m}\right)$ in *one* conversion period,

P amounts to $P\left(1 + \frac{j}{m}\right)^2$ in *two* conversion periods

[for $P\left(1 + \frac{j}{m}\right)$ at the beginning of the second period amounts to $P\left(1 + \frac{j}{m}\right) \cdot \left(1 + \frac{j}{m}\right)$ at the end. Here $A = P\left(1 + \frac{j}{m}\right)$]

P amounts to $P\left(1 + \frac{j}{m}\right)^3$ in *three* conversion periods

. . . .

P amounts to $P\left(1 + \frac{j}{m}\right)^{mn}$ in mn conversion periods, that is, in n years. Hence, when mn is an integer

$$S = P\left(1 + \frac{j}{m}\right)^{mn} \tag{6}$$

or, solving for P,
$$P = \frac{S}{\left(1 + \frac{j}{m}\right)^{mn}} = S\left(1 + \frac{j}{m}\right)^{-mn} \tag{6'}$$

In the derivation of formulas (6) and (6′) mn is an integer; these formulas, however, determine positive values for S and P when mn is not an integer. For example, when $P = 1$, $j = .06$, $m = 2$, $n = \frac{1}{4}$, formula (6) gives $S = 1.0148892$ approximately. In all cases, whether mn is integral or fractional, the positive value of S determined by (6) is called the *amount* or *accumulated value* of P for n years, and the positive value of P determined by (6′) is called the *present* or *discounted value* of S due in n years.

It should be noted that formula (6′), just as formula (2′), Art. 3, for discounting S, does not involve explicitly a rate of discount. Formula (2′) involves a simple interest rate, while formula (6′)

involves a compound interest rate. In Art. 6 it was seen that there is a definite simple discount rate which corresponds to a given simple interest rate. Likewise in Art. 13, it will be seen that there is a definite compound discount rate which corresponds to a given compound interest rate.

If
$$v = \frac{1}{\left(1 + \frac{j}{m}\right)} = \left(1 + \frac{j}{m}\right)^{-1}$$

then formulas (6) and (6′) can be written

$$S = Pv^{-mn} \tag{6}$$

$$P = Sv^{mn} \tag{6'}$$

If $\left(1 + \frac{j}{m}\right)$ be called an *accumulation factor* and v be called a *discount factor*, formulas (6) and (6′) may be stated verbally in the form: *To find the amount of any sum for n years, or mn periods at the nominal rate j converted m times per year, multiply this sum by the appropriate accumulation factor raised to the power mn; to find the present value of any sum due in n years multiply this sum by the discount factor to the same power.* If $m = 1$, the accumulation factor is $(1 + i)$, and the discount factor is $(1 + i)^{-1}$.

Table III gives values of integral powers of the accumulation factor correct to eight decimals. Table IV gives values of integral powers of the discount factor. Table VIII gives the values of fractional powers of the accumulation factor. Values of powers not found in these tables can be computed by use of logarithms.

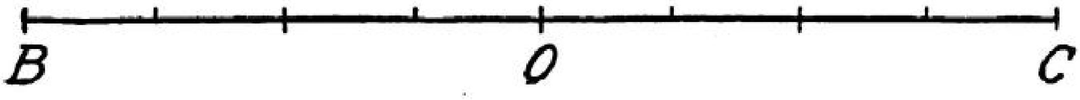
Formulas (6) and (6′) may be combined into a single formula by writing V in place of S or P on the left, and A in place of P or S on the right, and by writing t for both n and $-n$. This single formula is

$$V = A\left(1 + \frac{j}{m}\right)^{mt} \tag{7}$$

In formula (7), t is *positive* when the amount or accumulated value of A is to be found and *negative* when its present or discounted value is to be found. In other words, if A represents the value of a sum of money at a given time, formula (7) determines its value t

years later than this time when t is positive and its value t years earlier when t is negative.

Problems based on formula (7) can be visualized by the use of a diagram. In the following diagram each section of the line represents a half year:



The point O denotes any given time, any point to the right of O a later time, any point to the left of O an earlier time. By formula (7) the value at C of \$100 at O is $100\,(1.03)^4 = \$109.27$ if ($j = .06$, $m = 2$), and the value at B of \$100 at O is $100\,(1.03)^{-4} = \$88.85$ if the same rate is used. In the first instance $t = 2$; in the second $t = -2$.

In all similar diagrams in this book each point is associated with a definite time. Any point to the right of a fixed point O denotes a later time than that associated with O; any point to the left denotes an earlier time.

If I denotes the compound interest on P for n years, and D denotes the compound discount on S for n years, then by formulas (6) and (6′)

$$I = S - P = P\left[\left(1 + \frac{j}{m}\right)^{mn} - 1\right] \tag{8}$$

$$D = S - P = S\left[1 - \left(1 + \frac{j}{m}\right)^{-mn}\right] \tag{9}$$

In computing interest on P for a term n when mn is not integral, it is customary in many transactions to compute the compound interest on P for the term n' where mn' is the largest number of conversion periods contained in the term n and to add the simple interest on the amount of P at that time for the term $n - n'$; in computing discount on S for a term n when mn is not integral, it is likewise customary to compute the compound discount on S for the term n'' where mn'' is the smallest number of conversion periods containing the term n and to add the simple interest on the present value of S at that time for the term $n'' - n$.

EXAMPLE 1. Find the amount of \$1000 for a term of 4 years and 3 months at ($j = .06$, $m = 2$).

SOLUTION BY LOGARITHMS. By formula (6)

$$S = 1000\ (1.03)^{\frac{17}{2}} = \$1285.63$$

$$\begin{aligned} \log 1.03 &= 0.012837 \\ \log 1000 &= 3.000000 \\ \log (1.03)^{\frac{17}{2}} &= 0.109115 \\ \log S &= 3.109115 \end{aligned}$$

EXERCISE 1. Compute S by use of Tables III and VIII. [Hint. $(1.03)^{\frac{17}{2}} = (1.03)^8(1.03)^{\frac{1}{2}}$.]

EXERCISE 2. Solve Example 1 by finding the amount of \$1000 for 4 years at compound interest, and then finding the amount of this sum for 3 months at 6% simple interest.

EXAMPLE 2. Find the present value of \$1000 due in 2 years and 3 months at ($j = .06$, $m = 4$).

SOLUTION. By formula (6′)

$$\begin{aligned} P &= 1000\ (1.015)^{-9} \\ &= \$874.59 \quad \text{(Table IV).} \end{aligned}$$

EXERCISE. Compute P by use of logarithms.

EXAMPLE 3. Find the amount of \$1000 for a term of 35 years at ($j = .06$, $m = 4$).

SOLUTION. By formula (6)

$$\begin{aligned} S &= 1000\ (1.015)^{140} = 1000\ (1.015)^{70}(1.015)^{70} \\ &= \$8039.81 \end{aligned}$$

EXERCISES

1. Find, without using the tables, the compound interest on \$100 at ($j = .05$, $m = 2$) if $n = 2$.

2. Same as Exercise 1 except that ($j = .06$, $m = 4$) and $n = 1$.

3. Interpret the following:

$$\begin{aligned} &(a)\ \ V = 1000\ (1 + .03)^{-6} \text{ when } m = 2, \\ &(b)\ \ V = 1000\ (1 + .02)^{8} \text{ when } m = 4. \end{aligned}$$

4. Use Tables III and IV to find the values of

(a) $(1.025)^{10}$	(c) $(1.01125)^{48}$	(e) $(1.0275)^{-30}$
(b) $(1.035)^{25}$	(d) $(1.0275)^{-100}$	(f) $(1.0125)^{-80}$

Give at least two interpretations for each expression, using appropriate values for j, m, and n.

5. In the following diagram each section of the line represents a half year: the point O denotes any given time; any point to the right of O, a later time; any point to the left of O, an earlier time.

If (j = .06, m = 2), what value does \$100 at O have at B; at C; at D; at E? What is the value of t in each case if formula (7) is used?

6. Find the numbers to fill the blanks:

S	P	D	I	j	m	n
	125.00			.05	4	6
	62.50			.07	2	$4\frac{1}{3}$
917.28				.06	1	$2\frac{1}{2}$
815.25				.05	4	3

7. Find the amount of \$1200 for $4\frac{1}{4}$ years, using first the compound interest rate (j = .05, m = 2) for the whole term; second, this compound interest rate for the first four years, and then the simple interest rate, .05, for one fourth of a year. Ans. \$1480.25; \$1480.36.

8. Find the present value of \$1200 due in $4\frac{1}{4}$ years, using first the compound interest rate (j = .05, m = 2) for the whole term; second, this compound interest rate for $4\frac{1}{2}$ years, and then the simple interest rate, .05, for one fourth of a year. Ans. \$972.81; \$972.88.

9. Evaluate $(1.03)^{123}$, using Table III and the identity,*

$$(1.03)^{123} = (1.03)^{61}\,(1.03)^{62}$$

10. Evaluate $(1.03)^{-120}$, using Table IV and the identity

$$(1.03)^{-120} = (1.03)^{-60}\,(1.03)^{-60}$$

11. Evaluate $100(1.04)^{\frac{5}{2}}$, using Tables III and VIII.

12. Evaluate $(1.025)^{-\frac{17}{2}} = (1.025)^{-9}\,(1.025)^{\frac{1}{2}}$, using Tables IV and VIII.

* In calculating powers of $(1 + i)$ not found in the table, due regard should be given to the error involved in the product. If n represents the exponent, it can be shown by methods established in the calculus that the error is a minimum for even values of n if

$$(1 + i)^n = (1 + i)^{\frac{n}{2}}(1 + i)^{\frac{n}{2}} \text{ is used,}$$

and for odd values of n if

$$(1 + i)^n = (1 + i)^{\frac{n-1}{2}}(1 + i)^{\frac{n+1}{2}} \text{ is used.}$$

10. Compound discount formulas. By Art. 8, 1 at the beginning of any discount conversion period discounts to $\left(1 - \frac{f}{m}\right)$ at the end of the period, and hence A at the beginning discounts to $A\left(1 - \frac{f}{m}\right)$ at the end. It follows that

S discounts to $S\left(1 - \frac{f}{m}\right)$ in *one* period,

S discounts to $S\left(1 - \frac{f}{m}\right)^2$ in *two* periods,

. . . .

S discounts to $S\left(1 - \frac{f}{m}\right)^{mn}$ in mn periods, that is, in n years. Hence, when mn is an integer,

$$\boldsymbol{P = S\left(1 - \frac{f}{m}\right)^{mn}} \tag{10}$$

or, solving for S, $$\boldsymbol{S = \frac{P}{\left(1 - \frac{f}{m}\right)^{mn}} = P\left(1 - \frac{f}{m}\right)^{-mn}} \tag{10'}$$

In the derivation of formulas (10) and (10′) mn is an integer; these formulas, however, determine positive values for P and S when mn is not an integer. For example, when $S = 1$, $f = .06$, $m = 2$, $n = \frac{1}{2}$, formula (10) gives $P = 0.9848859$ approximately. In all cases, whether mn is integral or fractional, the positive value of P determined by (10) is called the *present* or *discounted* value of S due in n years, and the positive value of S determined by (10′) is called the *amount* or *accumulated value* of P for n years.

A discussion of the compound discount formulas may now be given which is entirely analogous to that given in Art. 9 of the compound interest formulas. This will not be given, however, since most of the transactions in finance are based on the principles of compound interest rather than on those of compound discount. An important application of the compound discount formula is found in the constant percentage or compound discount method of computing depreciation (Art. 63, Chapter III).

EXERCISES

1. Find the value at the end of 5 years of a house which cost \$6000 and which depreciated 3% each year. Use formula (10). Ans. \$5152.40.

2. If the cost of a house was \$6000 and its value at the end of 5 years was \$5250, find the compound discount rate, f, if $m = 1$.

3. Solve formula (10) for each of the letters involved in it except m.

11. Graphs of the compound interest and compound discount formulas. When P, j, and m are known, formula (6) expresses the relation between S and n, and when S, f, and m are known, formula (10) expresses that between P and n. The graphs of the corresponding simple interest and simple discount relations were found in Art. 7 to be straight lines. The graphs of these relations, however, are not straight lines since the variable n occurs as an exponent; they are exponential curves.

Table III may be used in constructing graphs of the compound interest formula. In constructing graphs of the compound discount formula a table of corresponding values for P and n must be computed. Figure 2 shows the graphs of

$S = P\left(1 + \frac{j}{m}\right)^{mn}$ for $P = m = 1$, and $\frac{j}{m} = .03, .04, .05, .06, .07$, and $.08$ and of $P = S\left(1 - \frac{f}{m}\right)^{mn}$ for $S = m = 1$ and $\frac{f}{m} = .04, .05, .06, .10, .15$, and $.20$ with respect to the axes ON, OA. The values of the compound interest I are represented by the vertical segments above $O'N'$ and those of the compound discount D, by the vertical segments between $O'N'$ and the discount curves. For example, MP_1 represents the amount and M_1P_1 the compound interest on 1 for 10 years at an interest rate of 3% per year and MP_2 represents the present value and P_2M_1 the compound discount on 1 for 10 years at a discount rate of 4% per year.

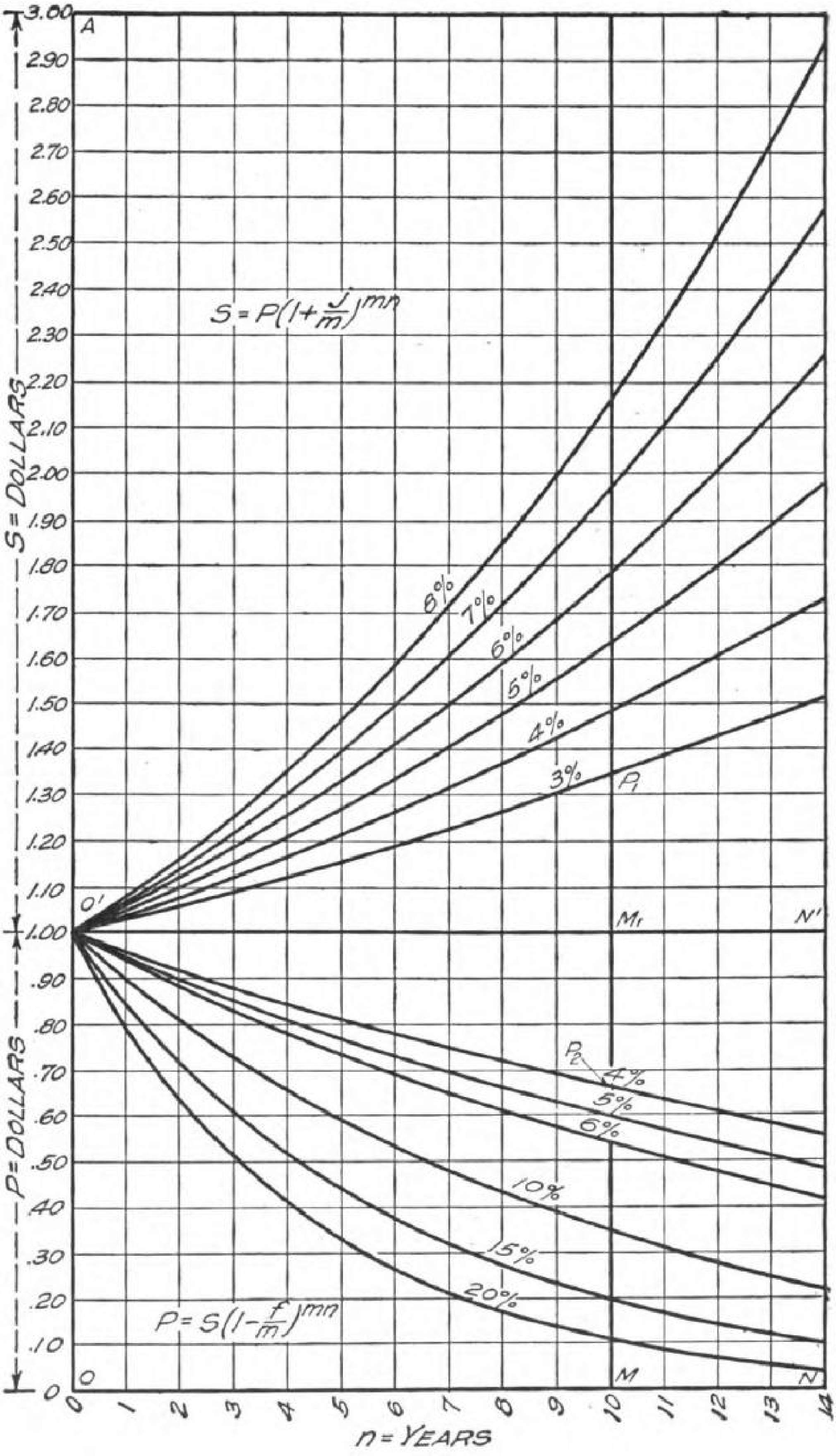

$S = P(1+\frac{j}{m})^{mn}$
$P = S(1-\frac{f}{m})^{mn}$
S = DOLLARS
P = DOLLARS
n = YEARS
3.00
2.90
2.80
2.70
2.60
2.50
2.40
2.30
2.20
2.10
2.00
1.90
1.80
1.70
1.60
1.50
1.40
1.30
1.20
1.10
1.00
.90
.80
.70
.60
.50
.40
.30
.20
.10
0
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
A
O'
O
M
N
M_1
N'
P_1
P_2
8%
7%
6%
5%
4%
3%
4%
5%
6%
10%
15%
20%

FIGURE 2

The values of P used in constructing the compound discount graphs shown in Figure 2 are given in the following table:

$$P = (1 - f)^n$$

n	$f = .04$	$f = .05$	$f = .06$	$f = .10$	$f = .15$	$f = .20$
1	.9600	.9500	.9400	.9000	.8500	.8000
2	.9216	.9025	.8836	.8100	.7225	.6400
3	.8847	.8573	.8306	.7290	.6141	.5120
4	.8493	.8145	.7807	.6561	.5220	.4096
5	.8153	.7738	.7339	.5905	.4437	.3277
6	.7827	.7351	.6899	.5314	.3772	.2621
7	.7514	.6983	.6485	.4783	.3206	.2097
8	.7214	.6634	.6096	.4305	.2725	.1678
9	.6925	.6303	.5730	.3874	.2370	.1342
10	.6648	.5987	.5386	.3487	.1969	.1074
11	.6382	.5688	.5063	.3138	.1673	.0859
12	.6127	.5403	.4759	.2824	.1422	.0687
13	.5882	.5133	.4474	.2452	.1209	.0550
14	.5646	.4877	.4205	.2288	.1028	.0440

The graphs in Figure 3 show the amounts of \$100 at 6% simple interest and at ($j = .06, m = 1$) compound interest for terms ranging from 0 to 25 years. These graphs show that the amounts at compound interest exceed those at simple interest for all terms greater than unity. For terms less than unity the amounts at simple interest exceed those at compound interest, as is seen in Figure 4.

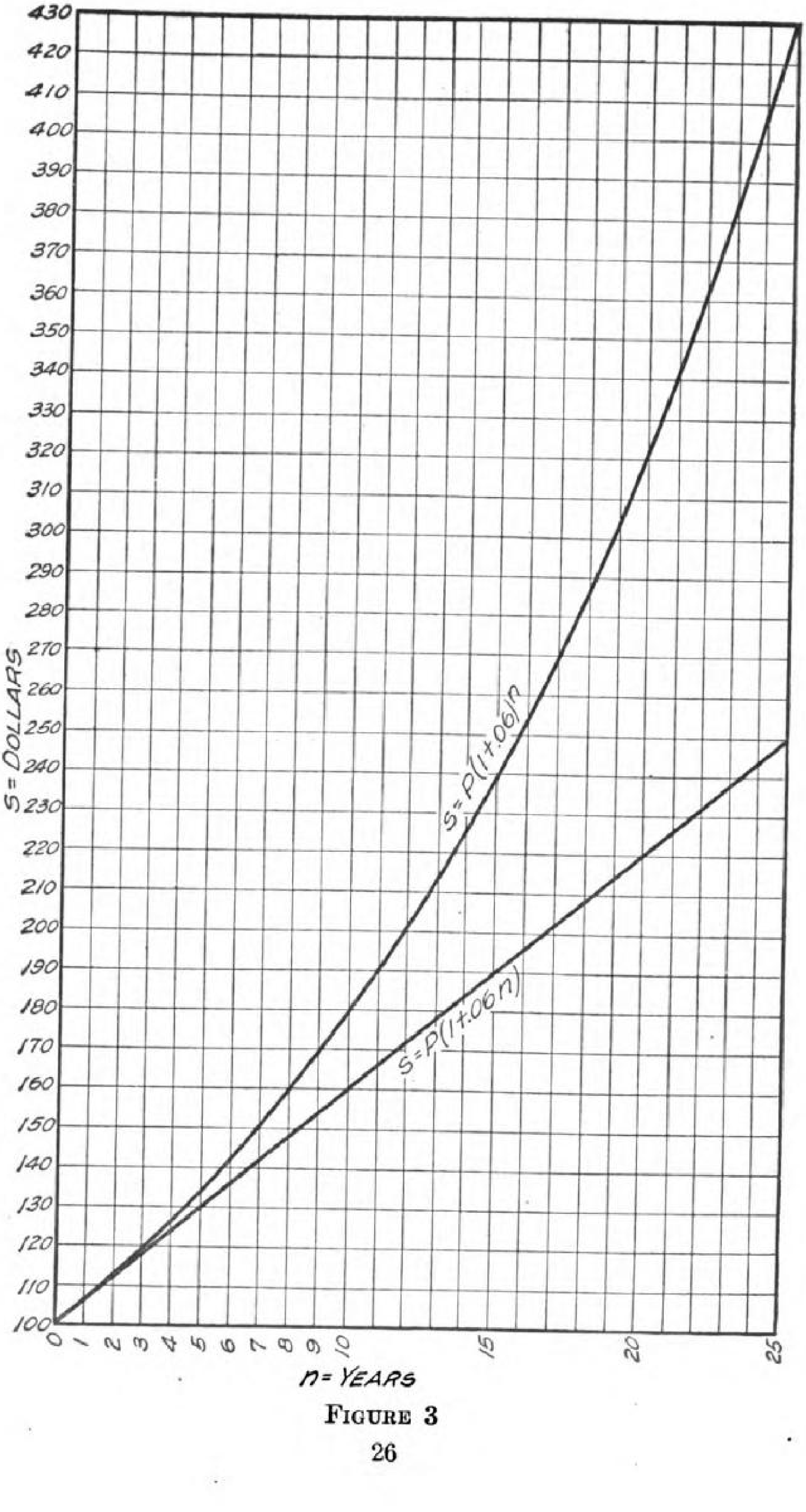


FIGURE 3

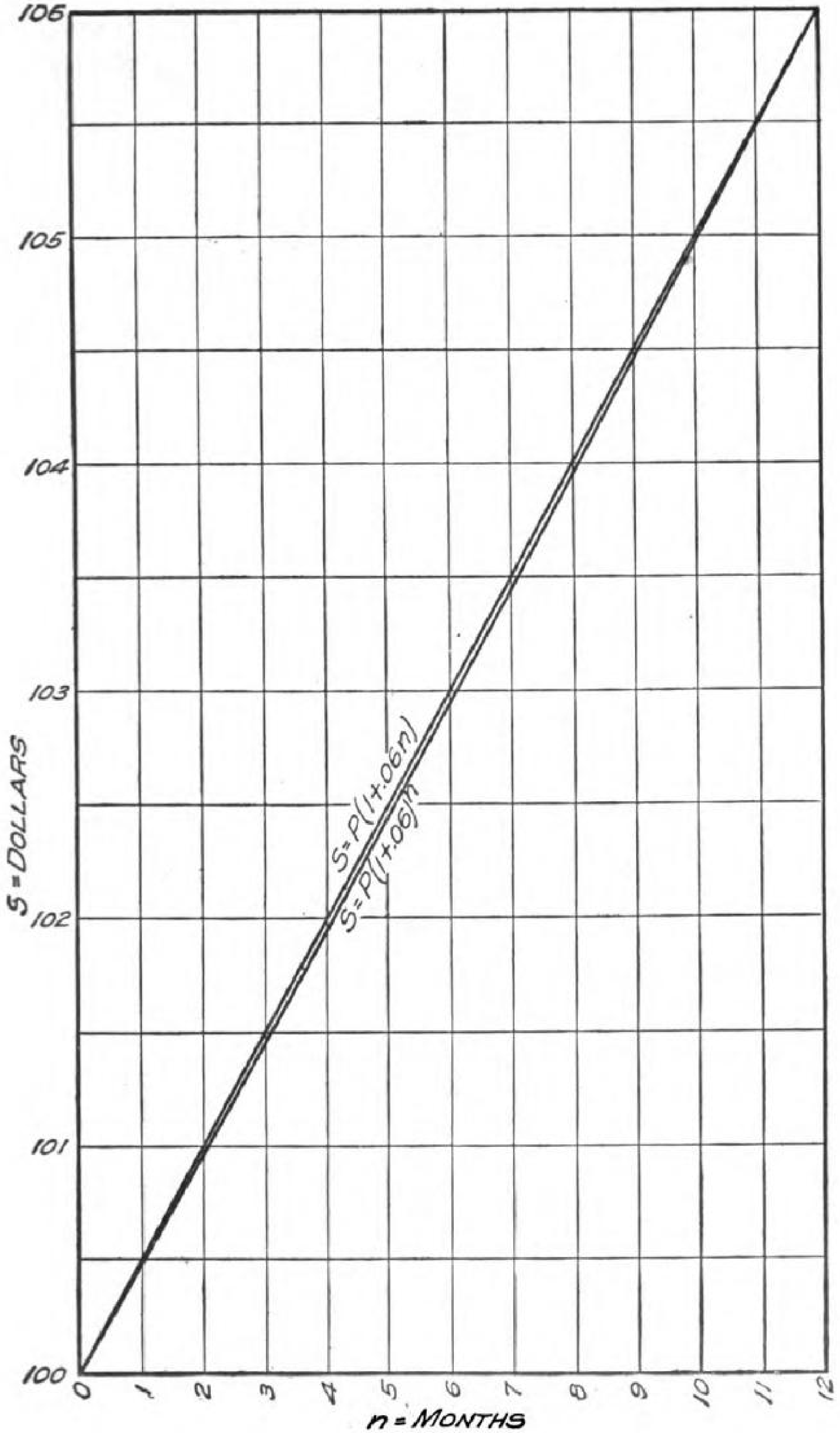

106
105
104
103
102
101
100
S = DOLLARS
S=P(1+.06n)
S=P(1+.06)^n
0
1
2
3
4
5
6
7
8
9
10
11
12
n = MONTHS

FIGURE 4

The values of S used in constructing the graphs in Figure 4 are given in the following table where $P = 100$, $j = .06$, and $m = 1$ and $i = .06$:

Term in Months	Compound Amount	Simple Amount
0	100.00	100.00
1	100.49	100.50
2	100.98	101.00
3	101.47	101.50
4	101.96	102.00
5	102.46	102.50
6	102.96	103.00
7	103.46	103.50
8	103.96	104.00
9	104.47	104.50
10	104.98	105.00
11	105.49	105.50
12	106.00	106.00

On account of the small scale used, the two graphs in Figure 3 seem to coincide for the range 0 to 1 year. In Figure 4 a sufficiently large scale is used to show the graphs distinct for this range.

EXERCISES

1. Construct the graph of $S = P\left(1 + \frac{j}{m}\right)^{mn}$ if $P = 100$, $\frac{j}{m} = 4\%$, and $m = 1$.

2. Construct the graph of $P = S\left(1 - \frac{j}{m}\right)^{mn}$ if $S = 100$, $\frac{j}{m} = 8\%$, and $m = 1$.

3. Construct the graph of $S = P\left(1 + \frac{j}{m}\right)^{mn}$ if $S = 110$, $P = 100$, and $m = 1$.

4. Construct the graphs of (*a*) $P = (1 - .06)^n$ and (*b*) $P = (1 - n\ .06)$.

5. On a large scale construct the graphs of the two equations in Exercise 4 for values of n by months for the first year. For what value of n do the curves intersect?

12. Problems based on the compound interest formulas. The compound interest formulas (Art. 9),

$$S = P\left(1 + \frac{j}{m}\right)^{mn} \tag{6}$$

$$I = P\left[\left(1 + \frac{j}{m}\right)^{mn} - 1\right] \tag{8}$$

contain six letters, and when four are known, the other two can be found by solving the two equations simultaneously for them. The solutions of these equations can be accomplished by the processes learned in elementary algebra together with the additional process of taking the logarithm of both members of an equation. Some of the computations that arise may be performed most easily by means of the fundamental operations of arithmetic, some by means of tables, and some by means of computing machines. The computations to be performed should be noted carefully before choosing the means to be used.

In Art. 9 some examples have been solved in which S or P is the unknown. In this article some examples will be solved in which n or j is the unknown.

Example 1. Find the term required for \$1000 to amount to \$3290.64 at $(j = .06, m = 4)$.

Solution. Formula (6) becomes, on dividing both members by 1000,

$$3.29064 = (1.015)^{4n}$$

Taking logarithms of both members and solving for n gives

$$n = \frac{\log 3.29064}{4 \log 1.015} = \frac{0.517280}{0.025864} = 20 \text{ years}$$

Exercise 1. Find the value of n by use of Table III.

Exercise 2. By taking logarithms of both members of formula (6) and solving for n show that

$$n = \frac{\log S - \log P}{m \log\left(1 + \frac{j}{m}\right)}$$

Example 2. Find the time required for any sum to double itself at $(j = .06, m = 1)$.

Solution. In this example $\frac{S}{P} = 2$, and formula (6) becomes $2 = (1.06)^n$.

Taking logarithms of both members and solving for n gives

$$n = \frac{\log 2}{\log 1.06} = \frac{0.301030}{0.025306} = 11.89 \text{ years}$$

EXERCISE. Find n by use of Table III.

EXAMPLE 3. If \$450 amounts to \$576.04 in 5 years, find j if $m = 2$.

SOLUTION. By formula (6) $576.04 = 450\left(1 + \frac{j}{2}\right)^{10}$

Taking logarithms and solving for $\log\left(1 + \frac{j}{2}\right)$ gives

$$\log\left(1 + \frac{j}{2}\right) = \frac{\log 576.04 - \log 450}{10} = 0.000724$$

$$1 + \frac{j}{2} = 1.025$$

$$j = .05$$

EXERCISE 1. Find j by means of Table III.

EXERCISE 2. By taking logarithms of both members of formula (6) and solving for $\log\left(1 + \frac{j}{m}\right)$, show that $\log\left(1 + \frac{j}{m}\right) = \frac{\log S - \log P}{mn}$

EXERCISE 3. By dividing both members of formula (6) by P and solving for $\frac{j}{m}$, show that

$$\frac{j}{m} = \sqrt[mn]{\frac{S}{P}} - 1$$

EXERCISES

1. Find the numbers to fill the blanks.

P	j	m	n	S
750	.05	2	2	
750	.055	4	2	
1000	.0475	2		2000.00
2500	.06	12		3755.85
2750		2	$4\frac{1}{2}$	3492.50
293.65		1	5	381.74
	.05	12	5	285.32
	.0425	1	2	962.35

2. The sum of \$100 is placed in a building and loan company to the credit of a child at the time of its birth. If the company's rate of interest is ($j = .04$, $m = 4$), how much will be due the child when it becomes 21 years old?

3. Would it be more to the advantage of a capitalist to loan money at $(j = .0575, m = 4)$ or at $(j = .06, m = 1)$?

4. The sum of \$2000 is placed in a savings bank whose interest rate is $(j = .045, m = 4)$. At the end of 3 years \$1000 is withdrawn. Find the amount in the bank 5 years from the date the deposit was made.

5. The sum of \$1000 is deposited in a savings bank whose rate is $(j = .05, m = 2)$. If at the end of 4 years \$500 is withdrawn and \$600 at the end of an additional 6 years, find the value of the account 15 years from the date of the deposit.

6. A willed \$4939.38 to a library. The will provided that the legacy be invested until it amounted to \$10,000, and thereafter the income on this amount be used to purchase books. The legacy was invested at $(j = .055, m = 2)$. In how many years did it accumulate to \$10,000? If the \$10,000 is invested at $5\frac{1}{4}\%$ with semi-annual payments of interest, and if the first interest payment each year is invested at 5% simple interest for six months, how much is available at the end of each year for the purchase of books? Ans. 13 years; \$556.88.

7. A buys a house for \$5600, pays one-half in cash and gives a mortgage note bearing the rate $(j = .065, m = 2)$ for the balance. If at the end of 4 years he pays \$3000, how much will be due on the note at the end of 5 additional years? Ans. \$848.75.

8. Find the amount to which \$2500 will accumulate in $4\frac{1}{4}$ years at $(j = .05, m = 2)$, first by using the compound interest formula for the whole term, and second by using this formula for the largest integral value of mn contained in the term and the simple interest formula for the remaining part. What is the difference between the two amounts? Why does the second method give the larger amount? Ans. \$3096.57; \$3096.77.

9. To what amount will \$1000 accumulate in $2\frac{1}{4}$ years if it is invested at $(j = .055, m = 2)$? Solve in two ways as in Exercise 8.

10. Discount \$3000 due in 2 years and 2 months at $(j = .06, m = 4)$ by methods analogous to those used in Exercise 8. Ans. \$2636.83; \$2636.90.

11. Find the present value of \$3500 for 3 years and 5 months at $(j = .06, m = 2)$ by each of the two methods of Exercise 10.

12. Find the nominal interest rate j at which \$1000 will amount to \$1590.55 in 10 years if $m = 2$; if $m = 4$.

13. If a war savings stamp was sold in the month of January, 1918, for \$4.12, show that at $(j = .04, m = 4)$ the stamp was worth \$5.00 January 1, 1923.

14. A treasury savings certificate was sold January 5, 1924, for \$800. It is redeemable at \$1000 in five years. Find j if $m = 4$.

15. If the population of a certain city increased from 100,000 to 150,000 in 10 years, and if it is assumed that the population increased according to the

compound interest law, find the annual rate of increase on the assumption that this rate was constant.

13. Corresponding rates. The definitions of corresponding rates given in Arts. 6 and 8 may be generalized as follows: *Any two rates are said to be corresponding for a given value of n when each leads to the same accumulated value of P in n years.* By means of this definition and the accumulation formulas, the equation connecting any two corresponding rates can be written at once. For example, the equation connecting the interest rate j converted m times per year and its corresponding effective rate, i, for n years is $P(1+i)^n = P\left(1+\frac{j}{m}\right)^{mn}$; dividing both members of this equation by P and extracting the nth root gives

$$1 + i = \left(1 + \frac{j}{m}\right)^m \tag{11}$$

Solving for j and writing $j_{(m)}$ in place of j to show the number of conversions, gives

$$j_{(m)} = m[(1+i)^{\frac{1}{m}} - 1] \tag{11'}$$

It may be noted that in this case the corresponding rates, i and j, are independent of n. Table IX gives the values of $j_{(m)}$ corresponding to various values of i and m, and Table III may be used to find the values of i corresponding to various values of j and m.

In a similar way if the interest rate j' converted m' times per year corresponds to the interest rate j'' converted m'' times per year, the definition leads to the equation

$$\left(1 + \frac{j'}{m'}\right)^{m'} = \left(1 + \frac{j''}{m''}\right)^{m''}$$

A few examples will now be solved to illustrate further the method of finding the rate corresponding to a given rate.

EXAMPLE 1. Find the interest rate j converted quarterly which corresponds to an interest rate of 6% converted annually.

SOLUTION. Equating accumulated values of P for n years, dividing by P, and extracting the nth root lead to

$$\left(1 + \frac{j}{4}\right)^4 = 1.06$$

SOLVING, using Table VIII, $j = .0587$ approximately.

EXAMPLE 2. Find the interest rate, i, converted annually which corresponds to an interest rate of 6% converted quarterly.

SOLUTION. Equating accumulated values leads to

$$1 + i = 1.015^4$$
$$i = .0614 \text{ approximately. (Table III)}$$

EXAMPLE 3. Find the discount rate f converted semi-annually which corresponds to an interest rate of 6% converted quarterly.

SOLUTION. Equating accumulated values leads to

$$\frac{1}{\left(1 - \frac{f}{2}\right)^2} = (1.015)^4$$
$$f = .0587 \text{ approximately. (Table IV)}$$

EXAMPLE 4. Find the effective discount rate d which corresponds to the nominal discount rate ($f = .08$, $m = 2$).

SOLUTION. Equating accumulated values leads to

$$1 - d = (1 - .04)^2$$
$$d = .0784$$

Corresponding rates are useful in comparing investments bearing different rates.

EXERCISES

1. Use the following table to find the values of i corresponding to $j = .06$, $m = 1, 2, 3, 4, 6, 12, 365, 1000$, and ∞:

m	$1 + i = \left(1 + \frac{.06}{m}\right)^m$
1	1.06000000
2	1.06090000
3	1.06120800
4	1.06136355
6	1.06152015
12	1.06167781
365	1.06183130
1000	1.06183462
∞	1.06183654

Verify the first six values of $(1 + i)$ by use of Table III, and the next two by use of the binomial formula. (The value of $1 + i$ when $m \to \infty$ is here included for the sake of completeness. See Art. 14 for a discussion of this case.)

2. Find the numbers to fill the blanks, using the formulas for corresponding rates:

$j_{(m)}$	i	m	d	$f_{(m)}$
.05		4		
		2	.04	
	.03	4		
		2		.035

3. Find the numbers to fill the blanks, using the formula

$$\left(1 + \frac{j'}{m'}\right)^{m'} = \left(1 + \frac{j''}{m''}\right)^{m''}:$$

j'	m'	j''	m''
.05	2		4
.05	2		1
	1	.06	2
	4	.07	12

Express verbally each of these four problems, using the solutions obtained. Notice that if $m' > m''$, then $j' < j''$.

4. If the effective interest rate of income from an investment is 5%, find the two equivalent nominal rates if $m = 2$ in one case and $m = 4$ in the other.

5. Find the present value of \$1239 due in 2 years if $(j = .05, m = 1)$.

6. Find the discounted value of \$1239 due in 2 years if the effective rate of discount is 4.7618%.

7. Show that two rates are corresponding for a given value of n when each leads to the same discounted value of S in n years. Why do Exercises 5 and 6 have the same answer?

8. Show by use of Exercise 7 that an effective rate of interest of 5% corresponds to an effective rate of discount of 4.7618%.

9. Find the discount rate converted quarterly which corresponds to $(j = .05, m = 2)$.

10. What effective interest rate corresponds to a discount rate of $(f = .05, m = 2)$?

11. Find the numbers to fill the blanks where j and f are rates corresponding to $i = \frac{6}{100}$ or $d = \frac{6}{106}$.

m	j	f
1	.06000000	.05660377
2	.05912603	.05742828
3	.05883847	
4	.05869538	.05784655
6	.05855287	
12	.05841061	
365	.05827356	
1000	.05827061	.05826721
∞	.05826891	.05826891

14. The interest and discount formulas when m becomes infinite. When the interest rate, j, remains constant and the number of conversion periods, m, increases, the value of $\left(1 + \frac{j}{m}\right)^m$ also increases. An illustration of this for $j = .06$ and $m = 1, 2, 3, 4, 6, 12, 365, 1000, \infty$ has been given in Exercise 1, Art. 13. This exercise shows further that as m increases from 1 to 1000, $\left(1 + \frac{j}{m}\right)^m$ increases from 1.06 to 1.06183462 approximately, and this comparatively small increase in $\left(1 + \frac{j}{m}\right)^m$ suggests that as m increases beyond limit, $\left(1 + \frac{j}{m}\right)^m$ approaches some definite number as a limiting value. It may be readily seen, in fact, that the limiting value of $\left(1 + \frac{j}{m}\right)^m$ when m becomes infinite, written $L_{m \to \infty}\left(1 + \frac{j}{m}\right)^m$, is e^j, where e, the base of the Napierian logarithms, has the approximate value 2.71828183. The expression, $\left(1 + \frac{j}{m}\right)^m$, may be written in the form $\left(1 + \frac{j}{m}\right)^{\frac{m}{j} \cdot j}$ and from this it follows that

$$L_{m\to\infty}\left(1+\frac{j}{m}\right)^m = L_{m\to\infty}\left(1+\frac{j}{m}\right)^{\frac{m}{j}\cdot j}$$
$$= L_{x\to\infty}\left(1+\frac{1}{x}\right)^{x\cdot j}, \text{ where } x = \frac{m}{j}$$
$$= e^j, \text{ since, by definition, } L_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

In like manner it can be shown that $L_{m\to\infty}\left(1-\frac{f}{m}\right)^m = e^{-f}$

Hence, when m becomes infinite, the compound interest and compound discount formulas can be written in the forms respectively:

$$S = Pe^{jn} \tag{6$_1$}$$
$$P = Se^{-fn} \tag{10$_1$}$$

When j or f is given and m becomes infinite, the interest or discount is said to be converted continuously, at this interest or discount rate. When interest or discount is converted continuously, it follows, from Art. 13, that the relations connecting the corresponding rates i, j and the corresponding rates d, f are

$$1 + i = e^j \text{ and } 1 - d = e^{-f}$$

If j and f are given, these equations determine the values of i and d. The value of i is the largest amount of interest that can be earned on \$1 in one year at a fixed rate j and a variable m which increases without limit. Likewise the value of d is the largest amount of discount on \$1 in one year at a fixed rate f, and a variable m which increases without limit. If i and d are given, these equations determine the corresponding values of j and f when the number of conversions per year becomes infinite. These values of j and f are denoted by δ and δ' respectively and they are called the *force of interest* and the *force of discount* which correspond to the given effective rates i and d. Using δ and δ' in place of j and f, the above relations become

$$e^\delta = 1 + i \text{ and } e^{-\delta'} = 1 - d$$

from which $\qquad \delta = \log_e (1 + i) \text{ and } \delta' = -\log_e (1 - d)$

When i and d are corresponding effective rates, they satisfy the relation $1 + i = \dfrac{1}{1 - d}$. On substituting the values of $1 + i$ and $1 - d$ in terms of δ and δ', this relation becomes

$$\therefore e^{\delta} = \frac{1}{e^{-\delta'}} = e^{\delta'}$$

It follows that when i and d are corresponding effective rates that $\delta = \delta'$ (Exercise 11, Art. 13).

Since the force of interest, δ, is the value of j when m becomes infinite that corresponds to a given effective interest rate i, it follows from the relation $j = m[(1 + i)^{\frac{1}{m}} - 1]$, (formula 11) that $\delta = L_{m \to \infty}\, m[(1 + i)^{\frac{1}{m}} - 1]$. Equating this value of δ to that given above gives

$$\delta = L_{m \to \infty}\, m[(1 + i)^{\frac{1}{m}} - 1] = \log_e (1 + i)$$

EXAMPLE 1. **Find the amount of $1000 for 10 years at a nominal rate of 4% converted continuously.**

SOLUTION. By formula (6_1)

$$S = 1000\, e^{.4} \qquad \log e = .434294$$
$$= \$1491.83 \qquad \log e^{.4} = .173718$$
$$\log 1000 = 3.000000$$
$$\log S = 3.173718$$

EXAMPLE 2. **Find the effective interest rate which corresponds to the force of interest of 6%.**

SOLUTION. By the formula

$$e^{\delta} = 1 + i$$
$$e^{.06} = 1 + i$$

Taking logarithms, $\log_{10} (1 + i) = .06 \log_{10} e = (.06)(.434294)$

$$= .02605$$
$$1 + i = 1.0618$$
$$i = .0618$$

EXAMPLE 3. **Find the force of interest which corresponds to an effective interest rate of 6%.**

SOLUTION. By the formula $e^{\delta} = 1 + i$

$$e^{\delta} = 1.06$$

Taking logarithms, $\delta \log e = \log 1.06$

$$\delta = \frac{\log 1.06}{\log e} = (.025306)(2.302585)$$
$$= .0583$$

(See Exercise 11, Art. 13.)

EXERCISE. Show that $L_{m \to \infty}\, m[(1 + .06)^{\frac{1}{m}} - 1] = .058269$

EXERCISES

1. Find the amount of \$1000 at 6% for 1 year if the interest is converted continuously. Ans. \$1061.84.

2. Same as Exercise 1, except that the rate is $4\frac{3}{4}$%.

3. What sum will amount to \$1000 in one year at $4\frac{1}{2}$% converted continuously. Ans. \$956.00.

4. Find the force of interest correct to 4 decimals corresponding to an effective rate of 4%, of $4\frac{1}{2}$%, of 5%. Ans. .0392; .0440; .0487.

5. Find the effective rate of interest correct to 4 decimals corresponding to a force of interest of 4%, of $4\frac{1}{2}$%, of 5%. Ans. .0408; .0460; .0512.

6. Find the rate converted continuously which corresponds to ($j = .06$, $m = 1$), to ($j = .06$, $m = 2$), to ($j = .06$, $m = 4$). Ans. .0582; .0591; .0596.

7. What is the discount on \$1000 for 1 year at ($f = .07$, $m \to \infty$).
Ans. \$67.60.

8. Find the force of discount correct to 4 decimals corresponding to an effective rate of discount of 4%, of 5%, and of 6%. Ans. .0408; .0513; .0618.

9. Interpret the following table:

n	$1 + .06\,n$	$(1 + .06)^n$	$e^{.06\,n}$
0	1.0000	1.0000	1.0000
$\frac{1}{12}$	1.0050	1.0049	1.0050
$\frac{2}{12}$	1.0100	1.0098	1.0101
$\frac{3}{12}$	1.0150	1.0147	1.0152
$\frac{4}{12}$	1.0200	1.0196	1.0202
$\frac{5}{12}$	1.0250	1.0246	1.0353
$\frac{6}{12}$	1.0300	1.0296	1.0305
$\frac{7}{12}$	1.0350	1.0346	1.0356
$\frac{8}{12}$	1.0400	1.0396	1.0408
$\frac{9}{12}$	1.0450	1.0447	1.0460
$\frac{10}{12}$	1.0500	1.0498	1.0513
$\frac{11}{12}$	1.0550	1.0549	1.0565
$\frac{12}{12}$	1.0600	1.0600	1.0618

Verify the entries for $n = \frac{7}{12}$.

10. The following inequalities hold for the amounts of P for n years:

$$n > 1,\ Pe^{in} > P(1+i)^n > P(1+ni)$$
$$n < 1,\ Pe^{in} > P(1+ni) > P(1+i)^n$$
$$n = 1,\ Pe^{in} > P(1+ni) = P(1+i)^n$$

Verify for $P = 1$, $i = .06$, and $n = 10$, $\frac{1}{12}$, and 1.

11. Show that the inequalities in Exercise 10 are special cases of the following inequalities:

$$n > \frac{1}{m},\ Pe^{jn} > P\left(1+\frac{j}{m}\right)^{mn} > P(1+nj)$$
$$n < \frac{1}{m},\ Pe^{jn} > P(1+nj) > P\left(1+\frac{j}{m}\right)^{mn}$$
$$n = \frac{1}{m},\ Pe^{jn} > P(1+nj) = P\left(1+\frac{j}{m}\right)^{mn}$$

15. The value of a set of sums. By the value at a given time of a set containing just one sum is meant the accumulated or discounted value of the sum to that time. By the value at a given time of a set containing more than one sum is meant the sum of the values of the separate sums of the set at that time. In the preceding articles interest and discount formulas have been developed for finding the value at any given time of one sum due at another time. By application of these formulas to the separate sums of any set the expression for the value of the set at a given time can be easily written. The work of computing the value of an expression of this kind may be rather laborious when the number of sums in the set is large. The study of methods of computing is important in all work in the mathematics of finance and it is especially important in finding the values of sets of sums, particularly of those sets which occur frequently in finance. Some sets of frequent occurrence consist of equal sums at periodic intervals during a definite term, as the dividend on bonds. Such sets are called *annuities certain*, and methods for finding their values are given in Chapter II. Other sets of frequent occurrence consist of sums at periodic intervals during a term not definitely known, as a pension payable during the life of some person. Sets of this type are called *contingent annuities*, and methods for finding their values are given in Chapter IV.

In writing the expression for the value of a set of sums it is necessary to know the formula to be used with each sum. *The*

compound interest formula is the one generally sanctioned by business practice and, except when otherwise stated, it will be assumed in what follows.

In finding the values at two or more times of a given set of sums by the compound interest formula, use will frequently be made of the following

Theorem I. *If V_o represents the value at a given time of a set of sums, and V_t represents the value of the set t years later than the given time, if t is positive, and t years earlier, if t is negative,*

$$V_t = V_o\left(1 + \frac{j}{m}\right)^{mt}$$

Proof: Let A denote any sum of the set when it is due and $A_{t'}$ represent its value at the given time where t' is the number of years from the time A is due to the given time. If t' is positive, $A_{t'}$ represents the amount of A for t' years; if t' is negative, $A_{t'}$ represents the present value of A due in t' years. If $A_{t'+t}$ represents the value of A, $t' + t$ years from the time it is due, then by formula (7)

$$\begin{aligned} A_{t'+t} &= A\left(1 + \frac{j}{m}\right)^{m(t'+t)} \\ &= A\left(1 + \frac{j}{m}\right)^{mt'} \cdot \left(1 + \frac{j}{m}\right)^{mt} \text{ by } a^{x+y} = a^x \cdot a^y \\ &= A_{t'}\left(1 + \frac{j}{m}\right)^{mt} \text{ by formula (7)} \end{aligned}$$

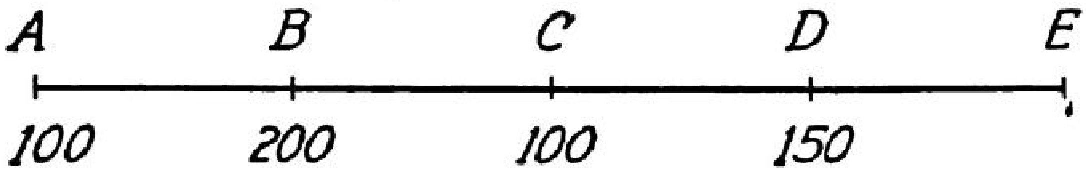
It follows that the theorem is true for each sum of the set, and hence it is true for the set as a whole. This theorem is also true when the compound discount formula at a fixed rate is used, but it is not true in general when the simple interest or simple discount formula is used.

So far in finding expressions for the value at a given time of a set of one or more sums the same rate has been assumed for the entire time during which the value of each sum is found. More general expressions for the value at a given time of a set of sums can be found by resolving the term, during which the value of each sum is found, into parts and using different rates with the separate parts. A case which is sometimes useful is that in which the

entire range over which values are to be found is divided into parts, each part having a constant interest rate for all sums whose values depend on it. For example, the amount of \$100 for 10 years at ($j = .06, m = 2$) during the first five years and at ($j = .07$, $m = 2$) during the last five is $100(1.03)^{10}(1.035)^{10}$. The above theorem can be extended to this case by using the appropriate interest rate for each part of the range.

EXERCISES

The diagram below, each section of which represents one year, shows a set of sums:



1. Find the value of this set at E at ($j = .06, m = 1$). Ans. \$635.81.

2. Find the value of this set at A at ($j = .06, m = 1$), first by discounting \$635.81 for four years, second by summing the discounted values at A of the separate sums. Ans. \$503.62.

3. Find the value of this set at C at ($j = .06, m = 1$), first by discounting \$635.81 for two years, second by accumulating \$503.62 for two years, third by summing the values at C of the separate sums.

4. Find the value of the set at E, using a simple interest rate of 6%.
Ans. \$631.90.

5. Find the value of this set at A, using a simple interest rate of 6%. Show that this value at A cannot be found by discounting \$631.00 for four years at the simple interest rate of 6%. Ans. \$505.08.

6. Find the value of the set at A if ($j = .05, m = 2$) for the first two years and ($j = .06, m = 2$) for the last two years.

16. Sets of sums having equal values. Equation of value. The equations needed to find the unknowns in solving problems in the mathematics of finance are usually gotten by equating the values of two sets of sums at a definite time. As seen in Art. 15, the expressions for values of the sets are found by use of the formulas for accumulating or discounting a single sum, and hence the accumulating and discounting operations are of primary importance in determining equations (Art. 1). Each of the interest or discount formulas expresses equality in value between the sums

S and P in the sense that each has the same value at a given time. For example, $S = P\left(1 + \frac{j}{m}\right)^{mn}$ states that the value of P when accumulated for n years at the interest rate j converted m times per year is S. In these formulas each set consists of a single sum. More general formulas or equations arise when the number of sums in one or both of the sets is greater than one. In all of these equations there are involved sums of money, rates, and terms of years, so that an unknown may be a sum of money, a rate, or a term of years.

In writing an equation expressing equality in value between two sets of sums, use may be made of the following

Theorem II. — *If two sets of sums are equal in value at a known time, they are also equal at any other time t years from the known time when the compound interest formula at a nominal interest rate j converted m times per year is used in finding values.*

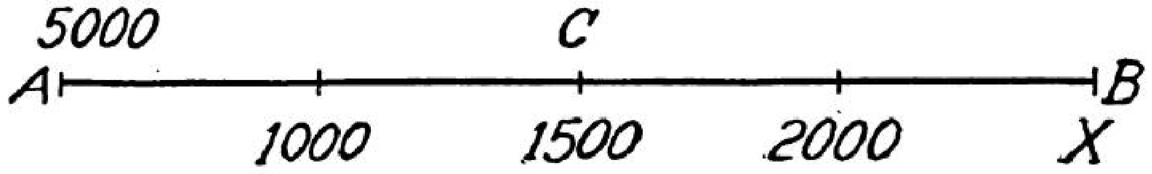
If V_0 is the value of either set at the known time, it follows from Theorem I that the value of either set t years from this time is V_t where $V_t = V_0\left(1 + \frac{j}{m}\right)^{mt}$; hence Theorem II is true. This theorem is also true when the compound discount formula is used, but it is not true for every value of n when the simple interest or simple discount formula is used. By virtue of this theorem it is permissible to select that time for equating the values of two sets of sums which leads to the simplest equation in the unknown to be found. Theorem II is also true for the special case considered at the end of Art. 15.

Equations of value which arise in elementary finance usually have a single sum for one of the two sets. Some examples based on sets of sums having equal values at a known time will now be solved.

EXAMPLE 1. A debt of \$5000 is to be paid in instalments, including principal and interest, as follows: \$1000 at the end of one year, \$1500 at the end of two, \$2000 at the end of three, and the balance at the end of four years. Find the last payment if unpaid principal accumulates at ($j = .06$, $m = 1$).

The diagram AB, each section of which represents one year, shows the two sets of sums having equal values; the set above the line represents the

debt, the set below the line the payments on the debt or the credits. The last payment is denoted by x.* In this diagram A denotes the present time; points to the right of A denote later times.



SOLUTION. The two sets of sums shown in the diagram have equal values at A and hence, by Theorem II, they have equal values at any other time. Equating values at B, that is, at the time the last payment is made, gives

$$x + 2000(1.06) + 1500(1.06)^2 + 1000(1.06)^3 = 5000(1.06)^4$$

Solving, using Table III, $x = \$1315.97$

EXERCISE 1. Find x by equating values at A.

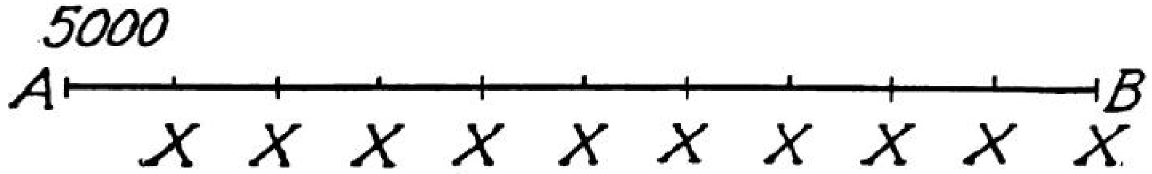
EXERCISE 2. Find x by equating values at C.

EXERCISE 3. Find x by computing, as in arithmetic, the unpaid principal of the debt just after each payment is made. This work may be arranged in a table or schedule as follows:

YEAR	PRINCIPAL AT BEGINNING OF YEAR	INTEREST AT ($j = .06$, $m = 1$)	PAYMENT AT END OF YEAR	PRINCIPAL REPAID
1	5000.00	300.00	1000.00	700.00
2	4300.00			
3				
4				

EXAMPLE 2. A debt of \$5000 is to be paid in ten equal semi-annual instalments, including principal and interest, the first to be paid at the end of six months. Find the amount of each instalment if unpaid principal accumulates at ($j = .06$, $m = 2$).

The diagram AB, each section of which represents a half year, shows the two sets of sums having equal values. Each instalment is denoted by x.



* Diagrams of this kind may be helpful to the learner in specifying both known and unknown data.

SOLUTION. Equating values at A gives

$$x\left(\frac{1}{1.03}+\frac{1}{(1.03)^2}+\cdots+\frac{1}{(1.03)^{10}}\right)=5000$$

Solving, using TABLE IV:

$$x = \$586.15$$

EXERCISE 1. Find x by equating values at B.

EXERCISE 2. Check the value found for x by constructing a schedule similar to that in Exercise 3, Example 1.

EXERCISE 3. Find the sum

$$\frac{1}{1.03}+\frac{1}{(1.03)^2}+\cdots+\frac{1}{(1.03)^{10}}$$

by use of the formula for summing a geometric progression.

EXAMPLE 3. Two non-interest-bearing notes, one for \$100 due in one year, the other for \$200 due in two years, are bought for \$264.06. Find the interest rate i converted annually for which the purchase price has the same value as the two notes.

SOLUTION. Equating the present value of the notes to 264.06 gives

$$\frac{200}{(1+i)^2}+\frac{100}{1+i}=264.06$$

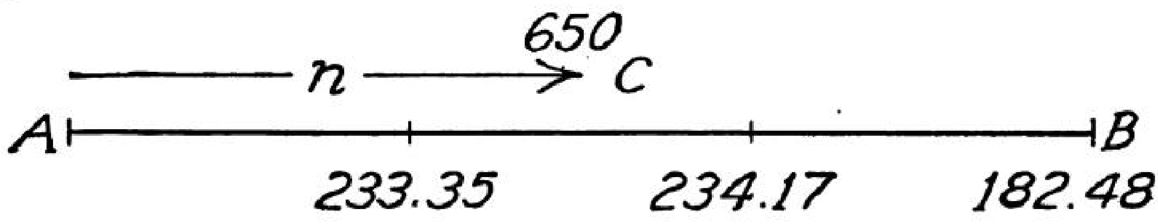
Solving, $i = .08$

EXERCISE 1. Check the value found for i by constructing a schedule.

EXERCISE 2. Solve Example 3, if a discount rate, d, converted annually, is used in place of the interest rate, i.

EXAMPLE 4. Three non-interest-bearing notes, one for \$233.35 due in 3 months, one for \$234.17 due in 6 months, and one for \$182.48 due in 9 months, are to be retired by a single payment equal in amount to the sum of three notes. On the basis of 6% simple discount when should this payment of \$650 be made if its present value equals the sum of the present values of the three notes?

The diagram AB shows the two sets of sums having equal values; n is the unknown time.



SOLUTION. Equating values at A gives

$$650[1-(.06)n] = 233.35[1-(.06)\tfrac{1}{4}] + 234.17[1-(.06)\tfrac{1}{2}] + 182.48[1-(.06)\tfrac{3}{4}].$$

Solving, $n = .48$ years, approximately.

EXERCISE 1. Solve Example 4 on the basis of 6% simple interest.

EXERCISE 2. Solve Example 4 on the basis of ($j = .06$, $m = 1$).

EXERCISE 3. Solve Example 4, if the two sets of sums have equal values at C.

It should be noted that the result found in Example 4 is independent of the discount rate used. When the same method is applied to a set of sums S_1 due in n_1 years, S_2 due in n_2 years, and so on, and to their sum $S_1 + S_2 \cdots$ there results

$$n = \frac{S_1n_1 + S_2n_2 + \cdots}{S_1 + S_2 + \cdots}$$

This formula gives a satisfactory means for finding the time at which the payment of a set of sums may be replaced by a single payment equal in value to their sum in case the numbers n_1, n_2, $\cdots$ denote short terms, each one year or less.

In the simple examples just solved, the unknown in each of the first two is a sum of money, in the third it is a rate, and in the fourth, a term of years. The method used to set up the equations is the same, however, in each case. Before giving other illustrations of this fundamental process, methods will be given for finding the values of sets of sums which occur at periodic intervals during a definite term of years, that is, of annuities certain. This is done in Chapter II.

EXERCISES

1. A man offers to sell his home for \$8500 cash or for \$1000 cash, \$2000 at the end of the first, \$3000 at the end of the second, and \$3600 at the end of the third year. On an interest basis of ($j = .06$, $m = 1$), how much better from the standpoint of the buyer is the cash price at the time of sale, at the end of the third year? Ans. \$79.41. \$94.58.

2. A washing machine is bought on the instalment plan for \$160. The contract of sale calls for a cash payment of \$20, and 10 monthly payments of \$14 each, the first of which is to be made one month from date of sale. Find the cash price on the day of sale if 6% simple discount is used. Ans. \$156.15.

3. A farm was purchased for \$12,000, \$3000 of which was paid in cash. The balance was paid in four equal annual instalments which began one year from date of purchase. On the basis of ($j = .05$, $m = 2$), find the amount of each instalment. Check by constructing a schedule. Ans. \$2541.79.

4. The cost of a piano is \$450 cash or \$125 cash and \$125 at the end of 3, 6, and 9 months. On the basis of 6% simple discount, how much better from the standpoint of the buyer is the cash price on the day of sale? Ans. \$38.75.

5. A man rents a house for \$600 per year payable in advance. What monthly rent payable in advance is equivalent at the beginning of the year to this annual rent? Use a 6% simple interest rate. Ans. \$51.33.

6. The sum of \$585 paid at the end of one year will retire two non-interest-bearing notes, one for \$300 due in 2 years, and the other for \$350 due in 3 years. If $m = 1$, find the interest rate correct to four decimal places.

Ans. $j = .0712$.

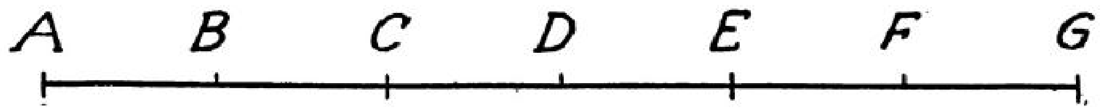
7. A owes B the sum of \$500. In partial payment A gives B a note drawn by C for \$300 due in one year without interest, and agrees to pay the balance in two equal annual instalments, the first of which is to be paid one year after the note matures. If an interest rate of $(j = .05,\ m = 2)$ is used, find the amount of each payment. Ans. \$121.28.

8. A owes B \$15,000. One year from date A pays B \$3000, and at the end of the second year \$4000. A then agrees to pay the balance in four equal semi-annual instalments, the first of which is to be paid 6 months after the \$4000 payment was made. On the basis of $(j = .06,\ m = 2)$ find the amount of each instalment. Check by constructing a schedule. Ans. \$2609.54.

9. A man owes two non-interest-bearing notes, one for \$150 due in 2 months, and one for \$300 due in 3 months. When must the sum of \$445 be paid if its present value equals that of the two sums? When must the sum of \$450 be paid? Use a simple discount rate of 7%. Which result is independent of the rate?

10. Four non-interest-bearing notes, one for \$125 due in 60 days, one for \$100 due in 90 days, one for \$75 due in 30 days, and one \$125 due in 10 days, are retired by a single payment. If this payment is \$425 and is the equivalent of the four notes at the present time, when must it be made? If the payment is \$430, when must it be made? Use a 5% simple discount rate.

The following exercises have reference to the diagram given below in which each section represents 6 months.



11. Show that:

(*a*) \$100 at A, \$150 at C, and \$200 at D compose a set of sums which is equivalent to a set consisting of the single sum of \$487.22 at G, if $(j = .04,\ m = 2)$.

(*b*) The value at B of each set in 11 (*a*) is \$441.29.

(*c*) The values of the two sets in 11 (*a*) are equal at F.

(*d*) The values of the two sets in 11 (*a*) are not equal at B on the basis of 4% simple interest, or simple discount.

12. Show that:

(*a*) A set of \$100 at B and \$150 at D is equivalent to a set consisting of the single sum \$265.50 at E on the basis of 6% simple interest.

(*b*) On a basis of 6% simple interest, the two sets in 12 (*a*) are not equivalent at F. Find the simple interest rate at which the two sets in 12 (*a*) are equivalent at F.

13. If a set of sums consisting of \$100 at B and \$400 at E is equivalent to a single sum \$492.7754 at D, find j if $m = 2$.

14. How long after A must the sum of \$600 be paid if the set consisting of this single sum is equivalent at A to a set consisting of \$300 at C, \$200 at E, and \$100 at F. Use a simple discount rate d. Interpret the cancellation of d.

15. Same as Exercise 14 except that the two sets are equivalent at G, and a simple interest rate of i is used. Interpret the cancellation of the i.

16. Same as Exercise 14 except that the first set consists of the single sum of \$650.

17. If a set of sums consisting of x at B and y at D is equivalent to a set z at E on the basis of a simple interest rate i, show that they are not equivalent at C unless x and y satisfy the equation $y = (2 + i)x$.

CHAPTER II

ANNUITIES CERTAIN

17. Annuities. An annuity is a set of sums, usually equal in value, paid at periodic intervals. The word *annuity* implies that the interval is one year; but, as used, the interval may be any fractional or integral multiple of a year. The value of each sum is called the *rent*, the interval between successive sums the *rent period*, and the time between the *beginning* of the first rent period and the *end* of the last, the *term* of the annuity. The sum of the rent payments in one year is called the *annual rent*.

Annuities are classified with respect to their terms. When the term is definite, the annuity is called an *annuity certain;* when the term becomes infinite, it is called a *perpetuity*. When the term is indefinite, depending on some contingency such as the life of a person who receives the rent, the annuity is called a *contingent annuity*. The weekly wages of workmen, the monthly rentals on property, the semi-annual taxes on real estate, the annual interest payments on a sum of money each for a fixed term are illustrations of annuities certain.

Annuities are classified also with respect to the time in the rent period when the rent is paid. When the rent is paid at the beginning of its period, the annuity is called an *annuity due;* when paid at the end of its period, the annuity is called an *annuity immediate*, or more briefly an *annuity*.

Classifications of annuities based on the times when their terms begin, and on the value of the rent, will be given in Arts. 27 and 32 respectively. In what follows in this chapter the word *annuity* will mean an annuity certain.

In this chapter the fundamental formulas for finding the values of annuities certain are developed, and methods of solving problems based on these formulas are discussed. In deriving these formulas, use is made of the formula for summing a geometric progression.

EXERCISES

1. If, in a geometric progression, the first term is represented by a, the last term by l, the constant ratio by c, the number of terms by n, and the sum by s, show that

$$l = ac^{n-1}$$
$$s = a\frac{c^n - 1}{c - 1}$$

What does this formula become if $c < 1$ and $n \to \infty$?

2. Given the geometric progression

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \cdots \text{ to } n \text{ terms}$$

Find the last term and the sum of the terms when $n = 10$. Find the sum when $n \to \infty$.

3. If the first term is 1, the constant ratio, $1 + .06$, and the number of terms, 6, find the sum.

4. Same as Exercise 3, except that the number of terms is n.

5. If $a = 1$, $c = (1.05)^{-1}$, and $n = 4$, find s.

6. If $a = 1$, $c = (1 + i)^{-1}$, and $n = 10$, show that $s = \dfrac{1 - (1 + i)^{-10}}{i}$.

7. If, in an arithmetic progression, the first term is represented by a, the last term by l, the common difference by d, the number of terms by n, and the sum by s, show that

$$l = a + (n - 1)d$$
$$s = \frac{(a + l)n}{2}$$

18. The value of an annuity at the end of its term. In this chapter R denotes the rent, n the number of years in the term, and r the number of years in the rent period of any annuity or of any annuity due. Unless otherwise stated, the compound interest formula at the rate j converted m times per year is used in finding values.

The value of any annuity at the end of its term, often called the *amount* of the annuity, will be denoted by V_n. In this article the formula for the amount of an annuity is derived by the use of the compound interest formula and the formula for finding the sum of a geometric progression. The derivation is given for two simple and important special cases which occur frequently in practice and then for the general case.

CASE 1. *The interest and the rent are paid annually; that is,* $m = r = 1$. In this case there are n rent periods in the term of

the annuity, and one interest conversion period in each rent period, so that the successive rent payments, beginning with the first, are accumulated for $n-1$, $n-2$, $\cdots$ 1, 0 interest conversion periods. By formula (6), Chapter I, the values of the rent payments at the end of the term are $R(1+i)^{n-1}$, $R(1+i)^{n-2}$, $\cdots$, $R(1+i)$, and R. Hence

$$V_n = R[1+(1+i)+(1+i)^2+\cdots+(1+i)^{n-1}]$$

The right-hand member is a geometric progression, having R for the first term, $(1+i)$ for the ratio, and n for the number of terms. Summing this progression gives

$$V_n = R\,\frac{(1+i)^n-1}{i}$$

CASE 2. *The interest and the rent are paid m times per year; that is,* $mr = 1$. In this case there are mn rent periods in the term of the annuity and one interest conversion period in each rent period, so that the successive rent payments, beginning with the first, are accumulated for $mn-1$, $mn-2$, $\cdots$ 1, 0 interest conversion periods. By formula (6), Chapter I, the values of the rent payments at the end of the term are

$$R\left(1+\frac{j}{m}\right)^{mn-1},\quad R\left(1+\frac{j}{m}\right)^{mn-2},\quad \cdots,\quad R\left(1+\frac{j}{m}\right),\quad \text{and}\quad R.$$

Hence

$$V_n = R\left[1+\left(1+\frac{j}{m}\right)+\left(1+\frac{j}{m}\right)^2+\cdots+\left(1+\frac{j}{m}\right)^{mn-1}\right]$$

The right-hand member is a geometric progression, having R for the first term, $\left(1+\frac{j}{m}\right)$ for the ratio, and mn for the number of terms. Summing this progression gives

$$V_n = R\,\frac{\left(1+\frac{j}{m}\right)^{mn}-1}{\frac{j}{m}}$$

GENERAL CASE. *The interest and the rent may be paid at the same or different times, and* $\frac{n}{r}$ *is an integer.* In this case there are $\frac{n}{r}$ rent pe-

riods in the term of the annuity and mr interest conversion periods in each rent period, so that the successive rent payments, beginning with the first, are accumulated for $mr\left(\frac{n}{r}-1\right)$, $mr\left(\frac{n}{r}-2\right)$, $\cdots$, mr, 0 interest conversion periods. By formula (6), Chapter I, the values of the rent payments at the end of the term are $R\left(1+\frac{j}{m}\right)^{mr\left(\frac{n}{r}-1\right)}$, $R\left(1+\frac{j}{m}\right)^{mr\left(\frac{n}{r}-2\right)}$, $\cdots$, $R\left(1+\frac{j}{m}\right)^{mr}$ and R. Hence

$$V_n = R\left[1+\left(1+\frac{j}{m}\right)^{mr}+\left(1+\frac{j}{m}\right)^{2mr}, \cdots +\left(1+\frac{j}{m}\right)^{mr\left(\frac{n}{r}-1\right)}\right]$$

The right-hand member is a geometric progression having R for the first term, $\left(1+\frac{j}{m}\right)^{mr}$ for the ratio and $\frac{n}{r}$ for the number of terms. Summing this progression gives

$$V_n = R\,\frac{\left(1+\frac{j}{m}\right)^{mn}-1}{\left(1+\frac{j}{m}\right)^{mr}-1} \qquad (1)$$

In the derivation of formula (1), $\frac{n}{r}$ is an integer; the formula, however, determines a positive value for V_n when $\frac{n}{r}$ is not an integer. In the applications in elementary finance $\frac{n}{r}$ is usually integral.

Formula (1) may be stated verbally in the form: *To find the value of an annuity at the end of its term divide the compound interest on R for the annuity term by the compound interest on* 1 *for one rent period.*

EXAMPLE. If \$100 are deposited at the end of each half year in a bank which pays 5% converted semi-annually, find the amount of the deposits at the end of 5 years.

SOLUTION. By formula (1)

$$V_5 = 100\,\frac{1.025^{10}-1}{.025}$$
$$= \$1120.34 \qquad \text{(Table III)}$$

EXERCISES

1. Find the sum of the geometric progression

$$1,\ \left(1+\frac{j}{m}\right),\ \left(1+\frac{j}{m}\right)^2,\ \left(1+\frac{j}{m}\right)^3.$$

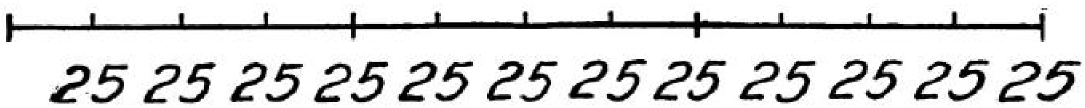
Evaluate this sum if $j = .06$ and $m = 2$.

2. Find the sum of the geometric progression

$$1,\ \left(1+\frac{.06}{4}\right),\ \cdots,\ \left(1+\frac{.06}{4}\right)^7$$

Compute the value of this sum by use of Table I.

3. The following diagram, each section of which denotes three months, represents the annuity having $R = 25$, $n = 3$, $r = \frac{1}{4}$:



Show that the amount of this annuity at ($j = .06$, $m = 4$) is given by

$$V_3 = 25[1 + 1.015 + (1.015)^2 + \cdots + (1.015)^{11}];$$

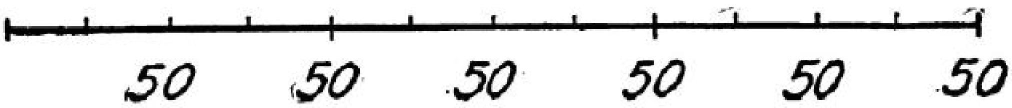
that the amount at ($j = .06$, $m = 2$) is given by

$$V_3 = 25[1 + (1.03)^{\frac{1}{2}} + 1.03 + \cdots + (1.03)^{\frac{11}{2}}]$$

Compute the value of V_3 in each case by use of the formula for the sum of a geometric progression, and by use of Tables III and VIII.

Ans. 326.03; 325.83

4. The following diagram, each section of which denotes three months, represents the annuity having $R = 50$, $n = 3$, $r = \frac{1}{2}$:



Show that the amount of this annuity at ($j = .06$, $m = 4$) is given by

$$V_3 = 50[1 + (1.015)^2 + (1.015)^4 + \cdots + (1.015)^{10}];$$

that the amount at ($j = .06$, $m = 2$) is given by

$$V_3 = 50[1 + 1.03 + (1.03)^2 + \cdots + (1.03)^5].$$

Compute the value of V_3 in each case by use of the formula for the sum of a geometric progression, and by use of Table III.

5. By formula (1) find the amount of an annuity having $R = 300$, $n = 10$, and $r = \frac{1}{4}$ at ($j = .05$, $m = 4$). Ans. 15446.87

6. By formula (1) find the amount of an annuity having $R = 1200$, $n = 10$, and $r = 1$ at ($j = .05$, $m = 2$). Ans. 15137.57

7. By formula (1) find the amount of an annuity having $R = 100$, $n = 10$, $r = \frac{1}{12}$ at ($j = .05$, $m = 4$).

8. The following diagram, each section of which denotes one year, represents the annuity having $R = 100$, $n = 5$, and $r = 1$:

Show that the amount of this annuity at 5% simple interest is

$$100[1.00 + 1.05 + 1.10 + 1.15 + 1.20];$$

that the amount of this annuity at $(j = .05, m = 1)$ is

$$100[1 + 1.05 + (1.05)^2 + (1.05)^3 + (1.05)^4]$$

Compute the value of each of these amounts; use the formula for the sum of an arithmetic progression in the first instance, that for a geometric progression in the second. Ans. 550; 552.56

9. At the simple interest rate i, show that the amount of the annuity whose rent is R, term n years, and rent period r years is

$$R\left(\frac{n}{r}\right)\left(1 + \frac{n - r}{2}i\right)$$

19. The value of an annuity at the beginning of its term. The value of an annuity at the beginning of its term, often called the *present value* of the annuity, will be denoted by V_o. The formula for the present value of an annuity can be derived by the method used in Art. 18 for deriving the formula for the amount. It can be derived more briefly, however, by applying Theorem I, Art. 15, Chapter I, to the value of V_n. By this theorem V_o is V_n discounted for n years; that is, $V_o = V_n\left(1 + \frac{j}{m}\right)^{-mn}$. Hence, using the value of V_n given by formula (1),

$$V_o = R\frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{\left(1 + \frac{j}{m}\right)^{mr} - 1} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Formula (2) may be stated verbally in the form: *To find the value of an annuity at the beginning of its term divide the compound discount on R for the annuity term by the compound interest on 1 for one rent period.*

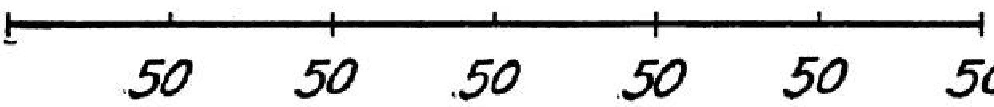
Example. Find the present value of an annuity whose rent is \$20, term is 3 years, and rent period is 1 month, at $(j = .06, m = 1)$.

SOLUTION. By formula (2)

$$\begin{aligned} V_o &= 20\,\frac{1 - 1.06^{-3}}{1.06^{\frac{1}{12}} - 1} \\ &= 20\,\frac{.16038072}{.00486755} \quad \text{(Tables IV and VIII)} \\ &= \$658.98 \end{aligned}$$

EXERCISES

1. The following diagram, each section of which denotes 6 months, represents the annuity having $R = 50$, $n = 3$, and $r = \frac{1}{2}$:



Show that the present value of this annuity at ($j = .06$, $m = 2$) is given by

$$V_o = 50[(1.03)^{-1} + (1.03)^{-2} + \cdots + (1.03)^{-5} + (1.03)^{-6}];$$

that the present value of this annuity at ($j = .06$, $m = 4$) is given by

$$V_o = 50[(1.015)^{-2} + (1.015)^{-4} + \cdots + (1.015)^{-10} + (1.015)^{-12}]$$

Compute the value of V_o in each case by use of the formula for the sum of a geometric progression, and by use of Table IV.

Ans. 270.86; 270.66

2. By formula (2) find the present value of the annuity having $R = 100$, $n = 5$, and $r = \frac{1}{2}$ at ($j = .05$, $m = 2$); at ($j = .05$, $m = 1$).

3. Show that the present value of the annuity of Exercise 1, at the simple discount rate $d = .06$ is

$$50[.97 + .94 + .91 + .88 + .85 + .82];$$

that the present value at the compound discount rate ($f = .06$, $m = 1$) is

$$50[(.94)^{\frac{1}{2}} + .94 + (.94)^{\frac{3}{2}} + (.94)^{2} + (.94)^{\frac{5}{2}} + (94)^{3}]$$

Compute the present value of each. Use the formula for the sum of an arithmetic progression in the first instance, that for the sum of a geometric progression in the second.

Ans. 268.50; 269.59

20. Annuity tables. Notation. When $R = r = 1$, V_n is denoted by $s_{\overline{n}|}$ and V_o is denoted by $a_{\overline{n}|}$. When $R = r = \dfrac{1}{p}$, V_n is denoted by $s_{\overline{n}|}^{(p)}$ and V_o is denoted by $a_{\overline{n}|}^{(p)}$. That is, $s_{\overline{n}|}$ denotes the amount and $a_{\overline{n}|}$ denotes the present value of an annuity whose rent is 1 payable annually for n years, and $s_{\overline{n}|}^{(p)}$ denotes the amount and $a_{\overline{n}|}^{(p)}$ denotes the present value of an annuity whose rent is $\dfrac{1}{p}$

payable p times per year for n years. In each of these annuities the annual rent is 1.

When m is greater than 1, the interest rate j converted m times per year may be indicated by writing (m) to the upper left of the symbol, and j to the lower right; when m is 1, it is omitted and j may be replaced by i.* For example,

${}^{(4)}s_{\overline{10}|.06}$ denotes the amount of an annuity whose rent is 1 payable annually for 10 years at ($j = .06$, $m = 4$);

$s^{(4)}_{\overline{10}|.06}$ denotes the amount of an annuity whose rent is $\frac{1}{4}$ payable quarterly for 10 years at ($j = .06$, $m = 1$);

$a_{\overline{10}|.06}$ denotes the present value of an annuity whose rent is 1 payable annually for 10 years at ($j = .06$, $m = 1$).

By formulas (1) and (2) and the notation just given it follows that

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \quad (1_1), \qquad a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i} \quad (2_1)$$

$$s^{(p)}_{\overline{n}|i} = \frac{1}{p}\,\frac{(1+i)^n - 1}{(1+i)^{\frac{1}{p}} - 1} \quad (1_2), \qquad a^{(p)}_{\overline{n}|i} = \frac{1}{p}\,\frac{1-(1+i)^{-n}}{(1+i)^{\frac{1}{p}} - 1} \quad (2_2)$$

Putting $n = \frac{1}{p}$ in (1_1) gives $s_{\overline{\frac{1}{p}}|i} = \frac{(1+i)^{\frac{1}{p}} - 1}{i}$; putting $n = 1$ in (1_2) gives $s^{(p)}_{\overline{1}|i} = \frac{1}{p}\,\frac{i}{(1+i)^{\frac{1}{p}} - 1}$. Hence, using formula (11′), Art. 13, Chapter I,

$$s^{(p)}_{\overline{1}|i} = \frac{1}{p s_{\overline{\frac{1}{p}}|i}} = \frac{i}{j_{(p)}} \qquad (1_3)$$

Table V gives the values, correct to eight decimals, of $s_{\overline{n}|i}$ for values of i and integral values of n which occur frequently in practice; Table VI gives analogous values of $a_{\overline{n}|i}$. It may be

* The notation here used is that of Glover in *Tables of Applied Mathematics in Finance, Insurance and Statistics* (Part I, page 3), published by George Wahr, Ann Arbor, Michigan.

readily verified that $\frac{1}{a_{\overline{n}|i}} = \frac{1}{s_{\overline{n}|i}} + i$. Table VII gives the reciprocals of the numbers in Table VI, that is, the values of $\frac{1}{a_{\overline{n}|i}}$. Table X gives the values of $s^{(p)}_{\overline{1}|i} = \frac{i}{j_{(p)}}$ for certain integral values of p. The tables just cited will be referred to as annuity tables.

EXERCISES

1. Interpret the meaning of the following symbols; draw a diagram in each case:

(a) $5\,s_{\overline{10}|.06}$

(b) $10\,a_{\overline{10}|.06}$

(c) $40\,s^{(4)}_{\overline{10}|.06}$

(d) $40\,s^{(4)}_{\overline{1}|.06}$

(e) $5^{(2)}s_{\overline{10}|.06}$

(f) $10^{(4)}a_{\overline{8}|.06}$

2. By use of Tables V, VI, VII, and X find the values of the following:

(a) $5\,s_{\overline{50}|.05}$

(b) $10\,a_{\overline{100}|.025}$

(c) $\dfrac{100}{s_{\overline{10}|.03}}$

(d) $\dfrac{50}{a_{\overline{25}|.045}}$

(e) $40\,s^{(4)}_{\overline{1}|.06}$

(f) $\dfrac{10}{s_{\overline{20}|.005}}$

21. New forms of the values of V_n and V_o. Formulas (1) and (2) can be put into forms well suited to computations with annuity tables by dividing numerator and denominator of the right-hand members by $\frac{j}{m}$ and then using formulas (1_1) and (2_1). This gives

$$V_n = R\frac{\dfrac{\left(1+\frac{j}{m}\right)^{mn}-1}{\frac{j}{m}}}{\dfrac{\left(1+\frac{j}{m}\right)^{mr}-1}{\frac{j}{m}}} \qquad V_o = R\frac{\dfrac{1-\left(1+\frac{j}{m}\right)^{-mn}}{\frac{j}{m}}}{\dfrac{\left(1+\frac{j}{m}\right)^{mr}-1}{\frac{j}{m}}}$$

$$V_n = R\frac{s_{\overline{mn}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}} \quad (3), \qquad V_o = R\frac{a_{\overline{mn}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}} \quad (4)$$

Formula (3) may be stated verbally in the form: *To find the amount of an annuity of rent R, term n, and rent period r, at the rate j converted m times per year, divide the amount of an annuity of rent R, term mn, and rent period 1 at the annual rate,* $\frac{j}{m}$, *by the amount of an annuity of rent 1, term mr and rent period 1 at the annual rate* $\frac{j}{m}$.

Formula (4) may be stated in an analogous form; this is left as an exercise.

When $mr = 1$, formulas (3) and (4) become, since $s_{\overline{1}|i} = 1$,

$$V_n = Rs_{\overline{mn}|\frac{j}{m}} \quad (3_1), \qquad V_o = Ra_{\overline{mn}|\frac{j}{m}} \quad (4_1)$$

When $mr = \frac{1}{p}$, formulas (3) and (4) become, since $\frac{1}{s_{\overline{\frac{1}{p}}|i}} = ps^{(p)}_{\overline{1}|i}$ by formula (1_3)

$$V_n = Rps_{\overline{mn}|\frac{j}{m}} \cdot s^{(p)}_{\overline{1}|\frac{j}{m}} \quad (3_2)$$

$$V_o = Rpa_{\overline{mn}|\frac{j}{m}} \cdot s^{(p)}_{\overline{1}|\frac{j}{m}} \quad (4_2)$$

Formulas (1), (2), (3), and (4) should be thoroughly mastered. Formulas (3) and (4) are adapted to computations based on annuity tables; formulas (1) and (2) are adapted to computations other than those based on annuity tables.

Example 1. If \$100 are deposited at the end of each half year in a bank which pays 5% converted semi-annually, find the amount of the deposits at the end of 5 years. (See Example, Art. 18.)

Solution. By formula (3_1)

$$V_5 = 100\, s_{\overline{10}|.025}$$
\$1120.34 (Table V)

Example 2. Find the present value of an annuity whose rent is \$20, term is 3 years, and rent period is 1 month, at ($j = .06$, $m = 1$). (See Example, Art. 19.)

Solution. By formula (4_2)

$$V_o = 240\, a_{\overline{3}|.06}\, s^{(12)}_{\overline{1}|.06}$$
$$= 240\,(2.67301195) \cdot (1.0272107) \quad \text{(Tables VI and X)}$$
$$= \$658.98$$

EXERCISES

1. Write the expressions for V_n and V_o by use of formulas (1) and (2), divide numerator and denominator by $\frac{j}{m}$, and write these results in the forms given by formulas (3) and (4), for each of the following:

R	n	r	j	m
100	10	1	.06	4
100	10	$\frac{1}{4}$	.06	2

2. Find V_n and V_o at (j = .06, m = 4), for each of the following:

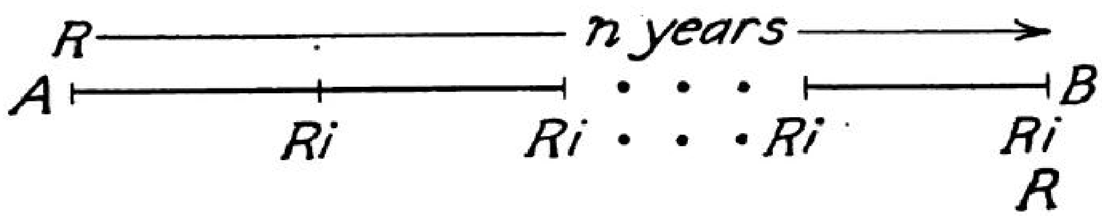
R	n	r	
100	2	$\frac{1}{4}$	Tables V and VI
10	8	$\frac{1}{2}$	Tables V, VI, and VII
1	2	$\frac{1}{12}$	Tables V, VI, and X

3. Write the expressions for the values of V_n and V_o, in the forms best suited for computations, for each of the following:

R	n	r	j	m
100	10	1	.055	4
100	10	1	.06	4
10	5	$\frac{1}{4}$	.05	2
10	5	$\frac{1}{52}$	.05	12
25	$2\frac{1}{6}$	$\frac{1}{12}$	.05	4
25	$\frac{1}{2}$	$\frac{1}{12}$	.04	1

22. Another derivation of the formulas for the values of V_n and V_o. The interest payments on R for n years at the rate i payable annually constitute an annuity of rent Ri and term n. The value of this annuity at the end of its term at the annual rate i is represented by iV_n and the value at the beginning of its term by iV_o.

It follows that, at this interest rate, the amount of R for n years is $R + iV_n$; likewise the present value of R due in n years is $R - iV_o$. The line diagram, AB, may be used to illustrate these statements:



In this diagram the R above is the original investment and the set of sums below is the return on this investment, so that the set below has the same value at any time as the R above.

Equating the values of these sets at B and A gives respectively

$$R + iV_n = R(1+i)^n \text{ from which } V_n = R\frac{(1+i)^n - 1}{i},$$

$$R(1+i)^{-n} + iV_o = R \text{ from which } V_o = R\frac{1-(1+i)^{-n}}{i}$$

An analogous treatment of the annuity composed of the interest payments on R for n years at the rate j payable m times per year leads to the equations

$$R + \frac{j}{m}V_n = R\left(1+\frac{j}{m}\right)^{mn} \text{ from which } V_n = R\frac{\left(1+\frac{j}{m}\right)^{mn} - 1}{\frac{j}{m}},$$

$$R\left(1+\frac{j}{m}\right)^{-mn} + \frac{j}{m}V_o = R \text{ from which } V_o = R\frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{\frac{j}{m}}$$

In general the compound interest amounts at the end of each r years on R for n years at the rate j converted m times per year, as given by formula (8), Art. 9, Chapter I, constitute an annuity, when $\frac{n}{r}$ is an integer, of rent $R\left[\left(1+\frac{j}{m}\right)^{mr} - 1\right]$, term n, and rent period r. An analogous treatment of this annuity at the rate j converted m times per year leads to the general equations

$$R + \left[\left(1+\frac{j}{m}\right)^{mr} - 1\right]V_n = R\left(1+\frac{j}{m}\right)^{mn}$$

from which
$$V_n = R\frac{\left(1+\frac{j}{m}\right)^{mn}-1}{\left(1+\frac{j}{m}\right)^{mr}-1} \tag{1}$$

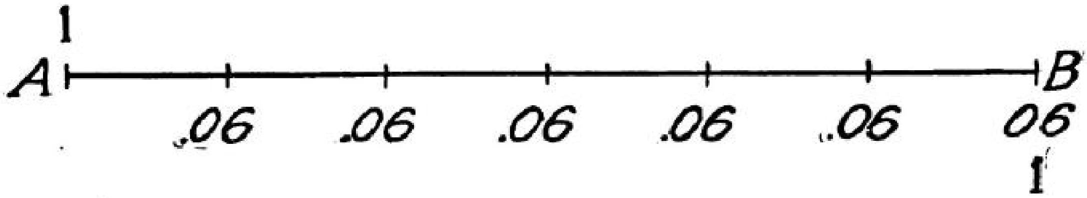
$$R\left(1+\frac{j}{m}\right)^{-mn}+\left[\left(1+\frac{j}{m}\right)^{mr}-1\right]V_o = R$$

from which
$$V_o = R\frac{1-\left(1+\frac{j}{m}\right)^{-mn}}{\left(1+\frac{j}{m}\right)^{mr}-1} \tag{2}$$

It may be noted that the derivations in this article do not use the formula for summing a geometric progression.

EXERCISES.

1. In the diagram each section of the line represents one year:



An investment of 1 is made at A at ($j = .06$, $m = 1$). The return on this investment is an annuity of .06 whose term is 6 years and in addition, 1 at B. Equating the values of the investment and of the return on the investment at B and at A gives respectively

$$(1.06)^6 = 1 + .06\, s_{\overline{6}|.06} \qquad \text{or} \quad s_{\overline{6}|.06} = \frac{(1.06)^6 - 1}{.06},$$
$$1 = (1.06)^{-6} + .06\, a_{\overline{6}|.06} \text{ or } a_{\overline{6}|.06} = \frac{1-(1.06)^{-6}}{.06}$$

2. By the use of a diagram similar to that in Exercise 1 show that

$$(1+i)^{-n} = 1 + i\, s_{\overline{n}|i} \qquad \text{or} \quad s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i},$$
$$1 = (1+i)^{-n} + i\, a_{\overline{n}|i} \text{ or } a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

23. Relations connecting $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$. The relations given in this article are of service in computation.

By Theorem I, $\qquad s_{\overline{n}|i} = a_{\overline{n}|i}(1+i)^n$

By formula (1_1), $\qquad s_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i}$

Solving each of these equations for $(1+i)^n$ and equating the results gives

$$\frac{s_{\overline{n}|i}}{a_{\overline{n}|i}} = 1 + i\, s_{\overline{n}|i}$$

Dividing both members of this equation by $s_{\overline{n}|i}$ gives the relation

$$\frac{1}{a_{\overline{n}|i}} = \frac{1}{s_{\overline{n}|i}} + i \qquad (5)$$

Relation (5) shows that the reciprocal of $a_{\overline{n}|i}$ is the reciprocal of $s_{\overline{n}|i}$, plus i, and that the reciprocal of $s_{\overline{n}|i}$ is the reciprocal of $a_{\overline{n}|i}$, minus i. Since i can be subtracted easily from $\frac{1}{a_{\overline{n}|i}}$ it follows that one table (Table VII) suffices for both $\frac{1}{a_{\overline{n}|i}}$ and $\frac{1}{s_{\overline{n}|i}}$.

To find the value of $s_{\overline{53}|.07}$ by means of Table V use may be made of the relation $s_{\overline{53}|.07} = s_{\overline{50}|.07}(1.07)^3 + s_{\overline{3}|.07}$. This relation may be readily verified by substituting the values of $s_{\overline{53}|.07}$, $s_{\overline{50}|.07}$, and $s_{\overline{3}|.07}$ given by formula (1_1). It may be derived by direct reasoning as follows: the annuity whose amount, $s_{\overline{53}|.07}$, is to be found can be resolved into two annuities, one composed of the first 50 rent payments and the other composed of the last three. The value of the first of these annuities at the end of 50 years is represented by $s_{\overline{50}|.07}$ and hence, by Theorem I, its value at the end of 53 years is $s_{\overline{50}|.07}(1.07)^3$. The value of the second annuity at the end of its term is represented by $s_{\overline{3}|.07}$. It follows that $s_{\overline{53}|.07} = s_{\overline{50}|.07}(1.07)^3 + s_{\overline{3}|.07}$. This same method of reasoning leads directly to the general relations

$$s_{\overline{n+n_1}|i} = s_{\overline{n}|i}(1+i)^{n_1} + s_{\overline{n_1}|i} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$a_{\overline{n+n_1}|i} = a_{\overline{n}|i}(1+i)^{-n_1} + a_{\overline{n_1}|i} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

When n and n_1 are found in the annuity tables but $n + n_1$ is not, these relations can be used to compute the amount and the present value of an annuity whose term is outside the range of the table. For example, $a_{\overline{120}|.06} = a_{\overline{60}|.06}(1.06)^{-60} + a_{\overline{60}|.06}$. In deriving formulas (6) and (7) the annuity whose term is $n + n_1$ years is resolved into two annuities whose terms are n and n_1 years. Resolving a given annuity into two or more components is frequently of service in computations.

When n_1 is 1, formulas (6) and (7) take the special forms, since $s_{\overline{1}|i} = 1$ and $a_{\overline{1}|i} = \dfrac{1}{(1+i)}$,

$$s_{\overline{n+1}|i} = s_{\overline{n}|i}\,(1+i) + 1 \tag{6'}$$

$$a_{\overline{n}|i} = a_{\overline{n+1}|i}(1+i) - 1 \tag{7'}$$

These special formulas may be used in constructing tables of $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$. For example, when n is 1, formula (6′) determines $s_{\overline{2}|i}$, then when n is 2, it determines $s_{\overline{3}|i}$, and so on. If $a_{\overline{50}|i}$ is computed from formula (3_1), then, when n is 49, formula (7′) determines $a_{\overline{49}|i}$, then when n is 48, it determines $a_{\overline{48}|i}$, and so on. In computing these tables by this process checks by direct computation should be made every 10 years or so to avoid errors.

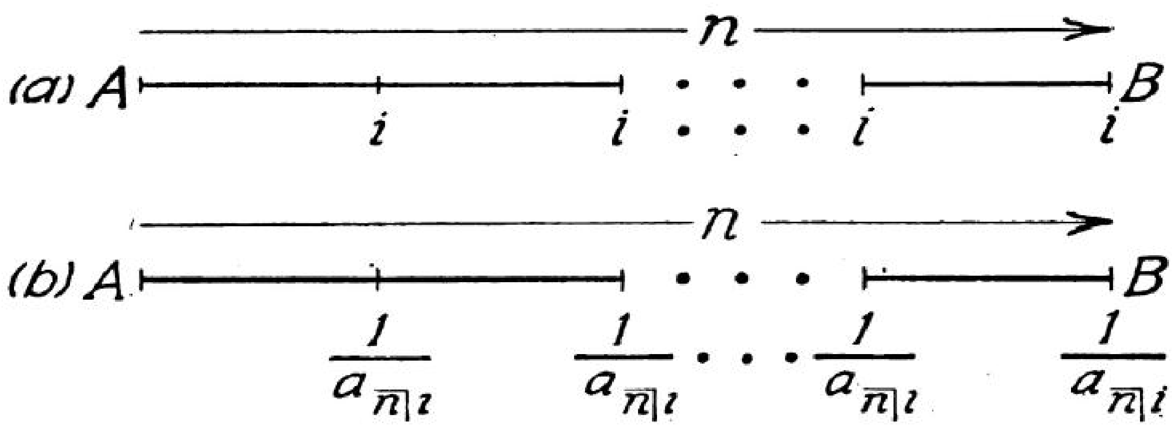
EXERCISES

1. Verify by use of Tables V and VI that $s_{\overline{5}|.06}$ and $a_{\overline{5}|.06}$ satisfy the equation

$$\frac{1}{a_{\overline{5}|.06}} = \frac{1}{s_{\overline{5}|.06}} + .06$$

2. Verify that $\dfrac{1}{a_{\overline{n}|i}} = \dfrac{1}{s_{\overline{n}|i}} + i$ by substituting $a_{\overline{n}|i} = \dfrac{1-(1+i)^{-n}}{i}$ and $s_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i}$.

3. Each section of the lines represents one year in the following diagrams:*



(1) Show that the sets of sums represented by these diagrams are equivalent at the interest rate i by showing that each is equivalent to 1 at A.

* Insert "1" under last i in diagram (a).

(2) Show that the value at B of the set of sums represented by diagram (a) is $i\,s_{\overline{n}|i} + 1$, and by diagram (b) is $\frac{s_{\overline{n}|i}}{a_{\overline{n}|i}}$.

(3) Equate the results in (2) and show that $\frac{1}{a_{\overline{n}|i}} = \frac{1}{s_{\overline{n}|i}} + i$.

4. Show that:

(a) $s_{\overline{5}|i} = s_{\overline{2}|i}(1+i)^3 + s_{\overline{3}|i} = s_{\overline{3}|i}\ (1+i)^2 + s_{\overline{2}|i} = s_{\overline{4}|i}(1+i)+1$

(b) $s_{\overline{n+1}|i} = s_{\overline{n}|i}(1+i)+1 = (1+i)^n + s_{\overline{n}|i}$

(c) $a_{\overline{5}|i} = a_{\overline{3}|i} + a_{\overline{2}|i}(1+i)^{-3} = a_{\overline{2}|i} + a_{\overline{3}|i}(1+i)^{-2} = (1+a_{\overline{4}|i})(1+i)^{-1} = a_{\overline{4}|i} + (1+i)^{-5}$

(d) $a_{\overline{n+1}|i} = (a_{\overline{n}|i}+1)(1+i)^{-1} = a_{\overline{n}|i} + (1+i)^{-(n+1)}$

5. Find the value of

(a) $s_{\overline{117}|.06}$ (Tables III and V)

(b) $a_{\overline{153}|.06}$ (Tables IV and VI)

(c) $\frac{1}{s_{\overline{117}|.06}}$

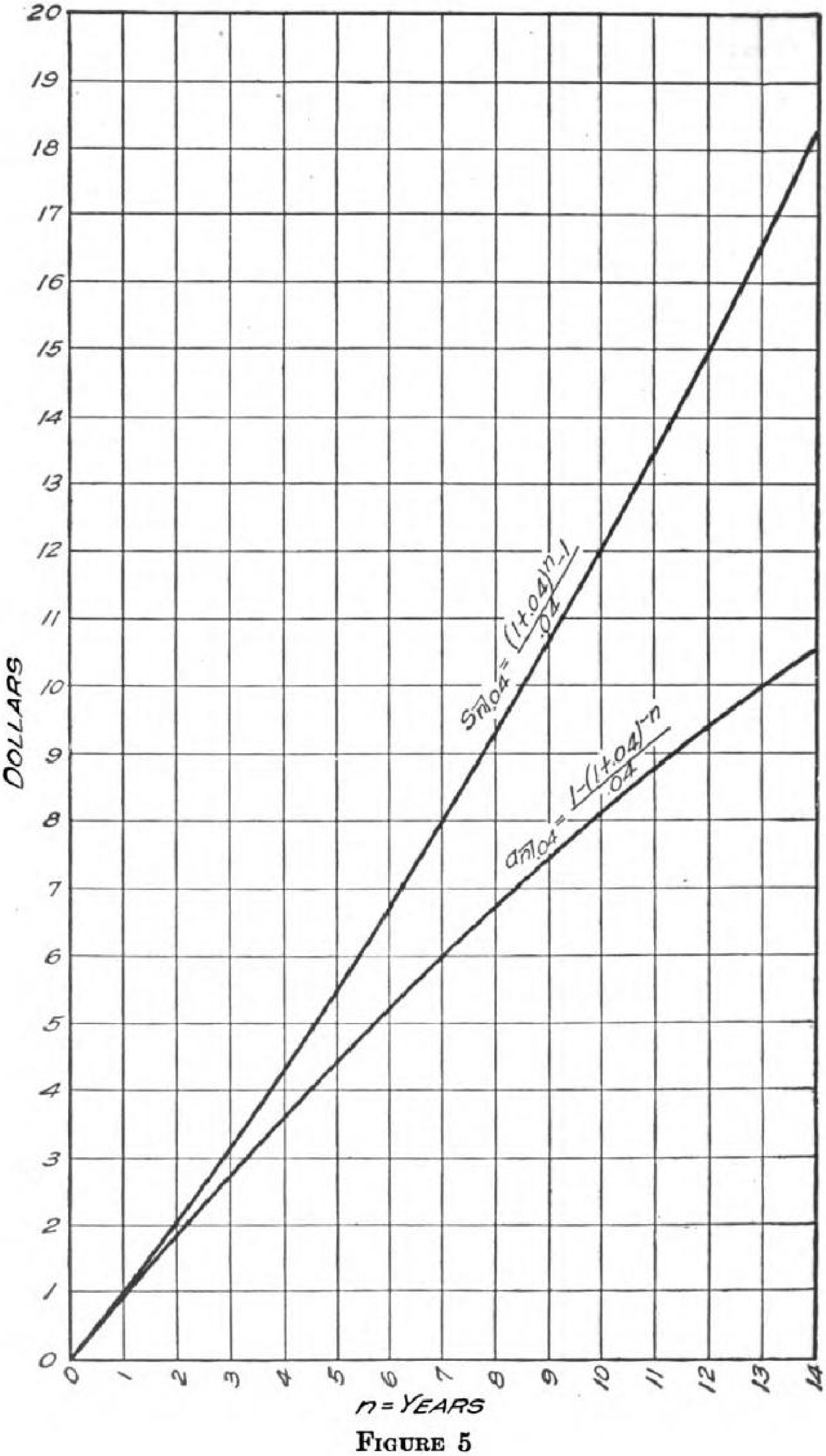
24. Graphs of $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$. Each of the formulas for the values of $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$ involves three letters and when one of these letters is given, each formula expresses a relation between the other two. These relations can be represented graphically.

Figure 5 shows the graphs of $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$ for $i = .04$ and $n = 0, 1, 2 \cdots 14$. Tables V and VI were used in constructing them. It may be noted that the slope of the $s_{\overline{n}|i}$ graph increases as n increases and that the slope of the $a_{\overline{n}|i}$ graph decreases as n increases.

Figure 6 shows the graphs of $s_{\overline{n}|i}$ and $a_{\overline{n}|i}$ for $n = 15$ and $i = .005, .01, 015 \cdots .07$.

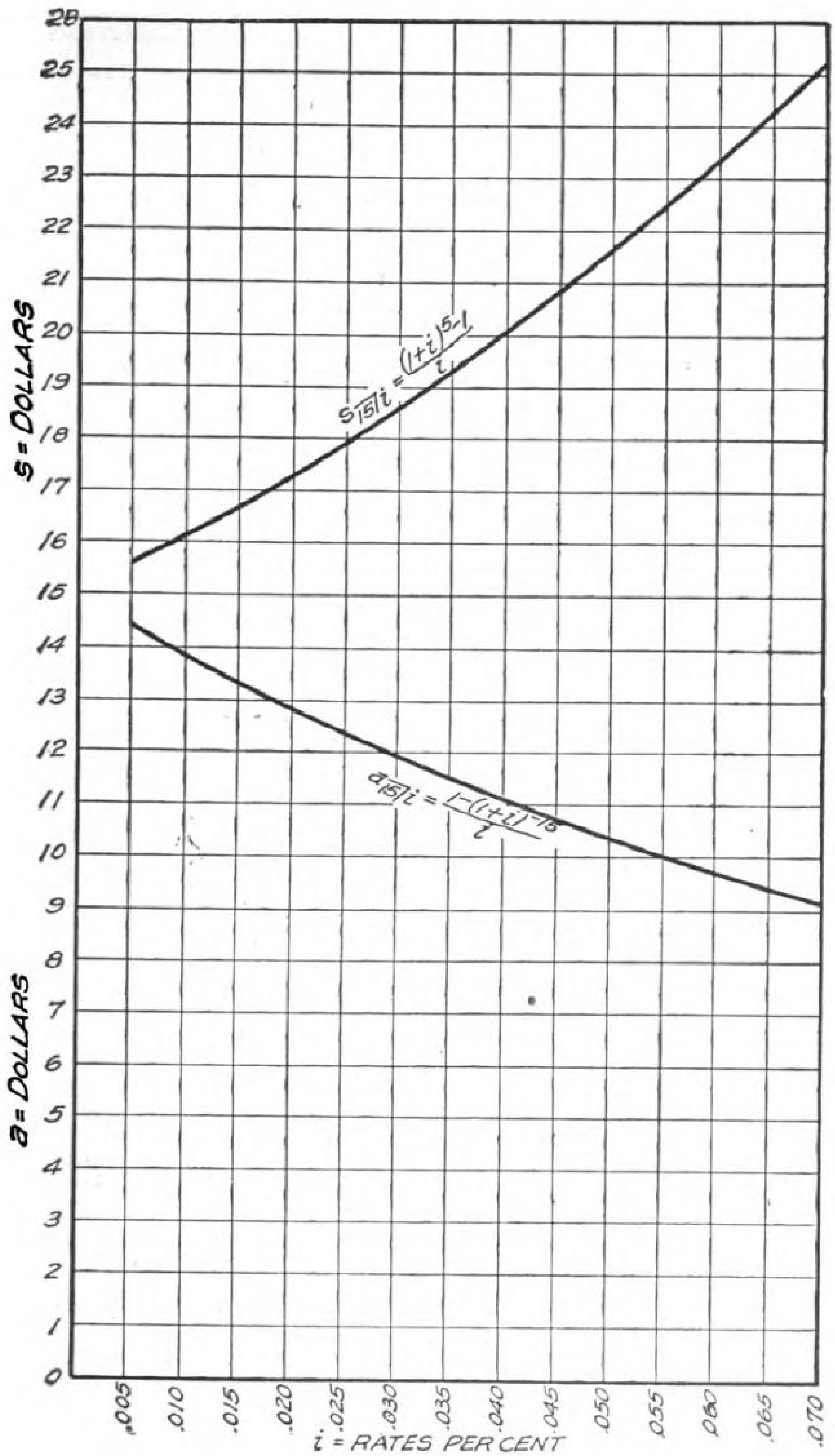
It may be noted that the segments of the graphs in Figure 6 which lie between adjacent values of i given in Tables V and VI are nearly straight lines.

Corresponding graphs of the more general functions V_n, V_o may also be constructed; when mr is different from one, it is necessary, however, to compute tables of values of V_n, V_o for assigned values of n or of i.



DOLLARS
n = YEARS
$s_{\overline{n}|.04} = \frac{(1+.04)^n - 1}{.04}$
$a_{\overline{n}|.04} = \frac{1-(1+.04)^{-n}}{.04}$

FIGURE 5



s = DOLLARS
a = DOLLARS
i = RATES PER CENT
$s_{\overline{15}|i} = \frac{(1+i)^{15}-1}{i}$
$a_{\overline{15}|i} = \frac{1-(1+i)^{-15}}{i}$

FIGURE 6

25. Problems based on the annuity formulas. Problems whose solutions are based directly on formulas (1), (2), (3), and (4) have V_n, V_o, R, n, or j for the unknown; m is given. The processes needed to find these unknowns are similar to those used in Art. 12, Chapter I, in solving problems based on the compound interest formula except when the interest rate is the unknown. In Arts 18, 19, and 21 some examples have been given in which V_n or V_o is the unknown. In this article some examples will be solved in which R or n is the unknown. The determination of the interest rate will be treated in Art. 26.

EXAMPLE 1. A man desires to accumulate \$4000 in 12 years by investing equal amounts at the end of each year. Find the annual investment if he can realize 4% converted quarterly.

SOLUTION. By formula (3)

$$4000 = R\frac{s_{\overline{48}|.01}}{s_{\overline{4}|.01}}$$

Solving,

$$\begin{aligned} R &= 4000\, s_{\overline{4}|.01} \frac{1}{s_{\overline{48}|.01}} \\ &= 4000(4.060401)(.01633384) \\ &= \$265.29 \end{aligned}$$

EXAMPLE 2. An investment of \$5000, bearing 4% payable annually is to be used, principal and interest, to repair a building at the end of each 5 years for 30 years. If the same amount is spent each time repairs are made, find this amount.

SOLUTION. By formula (4)

$$5000 = R\frac{a_{\overline{30}|.04}}{s_{\overline{5}|.04}}$$

Solving,

$$\begin{aligned} R &= 5000\, s_{\overline{5}|.04} \frac{1}{a_{\overline{30}|.04}} \\ &= 5000(5.41632256)(.0578301) \\ &= \$1566.13 \end{aligned}$$

EXERCISE. Check the value of R by constructing a schedule showing the outstanding principal at the end of each 5 years.

EXAMPLE 3. What payment made at the end of each six months for 3 years will discharge a debt of \$1200 at $(j = .06, m = 1)$.

SOLUTION. By formula (4)

$$1200 = R\frac{a_{\overline{3}|.06}}{s_{\overline{\frac{1}{2}}|.06}}$$

Solving,* $R = 1200\dfrac{s_{\overline{\frac{1}{2}}|.06}}{a_{\overline{3}|.06}}$

$$= 1200\frac{1}{a_{\overline{3}|.06}}\frac{(1+.06)^{\frac{1}{2}}-1}{.06}$$

$$= 20000(.37410981)(.02956302) \qquad \text{(Tables VII and VIII)}$$

$$= \$221.20$$

EXERCISE. Solve Example 3 by use of formula 2.

EXAMPLE 4. If instalments of $300 including principal and interest are paid at the end of each six months on a debt of $5000 bearing 6% payable semi-annually, find how many instalments are needed before the principal of the debt is less than $300. Also find the payment necessary to extinguish the debt at the end of the next six months.

SOLUTION. By formula (2)

$$5000 = 300\frac{1-(1.03)^{-2n}}{.03}$$

Solving,

$$1.03^{2n} = 2$$

$$2n = \frac{\log 2}{\log 1.03} = 23.44$$

This result indicates that 23 instalments of $300 each are needed to reduce the debt to less than $300. Let x be the additional payment at the end of 12 years which extinguishes the debt. Equating the present value of the payments to 5000 gives

$$\frac{x}{(1.03)^{24}} + 300\,a_{\overline{23}|.03} = 5000$$

Solving,

$$x = \$136.029$$

EXERCISE 1. Find x by equating the value of the payments at the time the 23d instalment is made to the value of the debt at that time.

EXERCISE 2. Find x by constructing a schedule showing the amount of the debt after each payment is made.

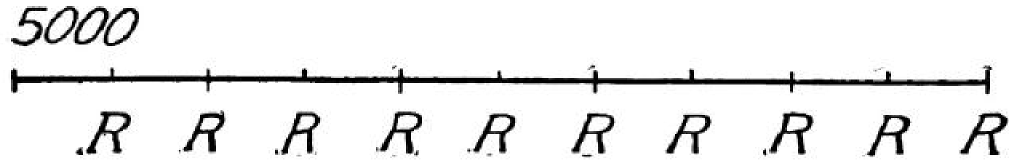
EXERCISE 3. Find $2n$ by interpolating in Table VI. (Hint. In this case $a_{\overline{2n}|.03} = 16.66666667$.)

EXERCISE 4. Solve $s_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i}$ for n in terms of i and $S_{\overline{n}|i}$

* Notice that both interest and annuity tables are used in this computation.

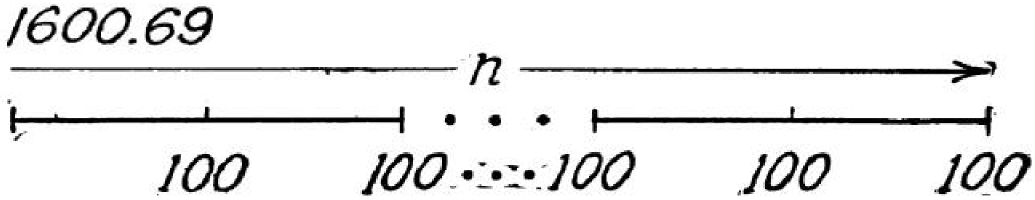
EXERCISES

1. In the following diagram each section of the line represents six months:



Find R if the set of sums consisting of the R's is equivalent to that consisting of the single sum of \$5000. Use $(j = .06, m = 2)$. Ans. \$586.15.

2. Each section of the line represents six months:



If $(j = .06, m = 1)$, find n if the two sets of sums are equivalent. Ans. 11 years.

3. A savings deposit of \$5 is made at the end of each week for 10 years. Find the amount of the deposits if $(j = .04, m = 4)$. Ans. \$3192.27.

4. A man buys a house for \$5500. He pays \$500 cash, and agrees to pay \$60 at the end of each month thereafter until the debt is paid. If the interest rate is $(j = .06, m = 12)$, how much will still be due at the end of four years? at the end of six years? How many payments will reduce the debt to less than \$60? If the debt is paid in full one month after it is reduced to less than \$60, find the last payment. Ans. \$3106.58; \$1975.69; 108; \$4.12.

5. The present value of an annuity of \$10 paid at the end of each six months for n years at $(j = .05, m = 2)$ is \$194.64. Find n, (*a*) by use of Table I, (*b*) by use of Table VI. Ans. 13.5 years.

6. The amount of an annuity of \$10 paid at the end of each six months for n years at $(j = .05, m = 2)$ is \$333.16. Find n, (*a*) by use of Table I, (*b*) by use of Table V. Interpret the meaning of the fractional value of n.

Ans. 12.27 years.

7. The rent of a house is \$60 a month. If the rent is to be paid at the end of each month, find the equivalent annual rent payable at the beginning of the year if $(j = .06, m = 1)$. Ans. \$697.73.

8. A debt of \$5000 is extinguished in 5 years by equal monthly payments. If $(j = .06, m = 4)$ and the payments are made at the end of each month, find the amount of each payment.

9. What is the cash price of a piano which is equivalent to \$50 cash and monthly payments of \$25 thereafter until \$500 in all is paid. Use $(j = .08, m = 4)$. Ans. \$472.19.

10. What sum deposited in a savings bank at the end of each month for 8 years will amount to \$1000, if (j = .04, m = 4). Ans. \$8.86.

11. A cash payment of \$3500 was made on a farm bought for \$10000. What payment made at the end of each year for six successive years paid the balance in full, if (j = .055, m = 2)? Ans. \$1304.28.

26. Methods for finding the interest rate. In this problem the equation to be solved is gotten by substituting the known values of V_n or V_o, n, r, and m into formulas (1) or (2). In practice, the value of j determined by one of these equations is usually an irrational number. There are two methods in common use for solving such equations. One of these is the *interpolation method;* and the other is *Newton's method.* For the accuracy needed in many problems of this type in finance, the interpolation method is preferable. A third method, well suited for solving rate equations, is the *method of iteration.**

In problems in which interpolation has been used heretofore to determine an unknown approximately, there has been just one interpolation for the unknown and the tables used for interpolating have been given; they have been logarithmic, interest and annuity tables. In problems in finance in which the rate is determined to a desired degree of accuracy by the method of interpolation more than one interpolation are often needed and the tables used for interpolating and for testing the accuracy of any approximation must be constructed. In a rate problem based on formula (1) or on formula (2), the relation between V_n or V_o and j, gotten by substituting the known values of m, n, and r into the one or the other of these formulas, can be used to construct such a table of values of V_n or V_o corresponding to assigned values of j.

Newton's method consists in replacing j in the equation to be solved by $j' + h$, where j' is any approximation already found and h is the correction or error, arranging the resulting equation in ascending powers of h, dropping the terms which contain powers of h greater than one, and solving for h. Then $j' + h$ is a closer approximation than j'. By repeating this process the root can

* See papers in Volume XXXII, 1925, of the *American Mathematical Monthly* by C. H. Forsythe, March Number, page 126, and by L. R. Ford, June–July Number, page 272.

be found to any desired degree of accuracy. The initial value of j' is usually gotten by selecting one of two numbers between which the root lies in a table constructed as in the method of interpolation, by selecting the approximation found by one interpolation in such a table, or by use of some approximation formula.

By means of a graph the geometric significance of an interpolation and also of an application of Newton's process may be seen. Let $f(x) = k$, where k is a constant, be any equation having a root, x, between the two numbers a and b, and let Figure 7 represent the graph of $y = f(x)$ from a to b inclusive, MP being the ordinate of length k, OD being the abscissa a, and OE the abscissa b.

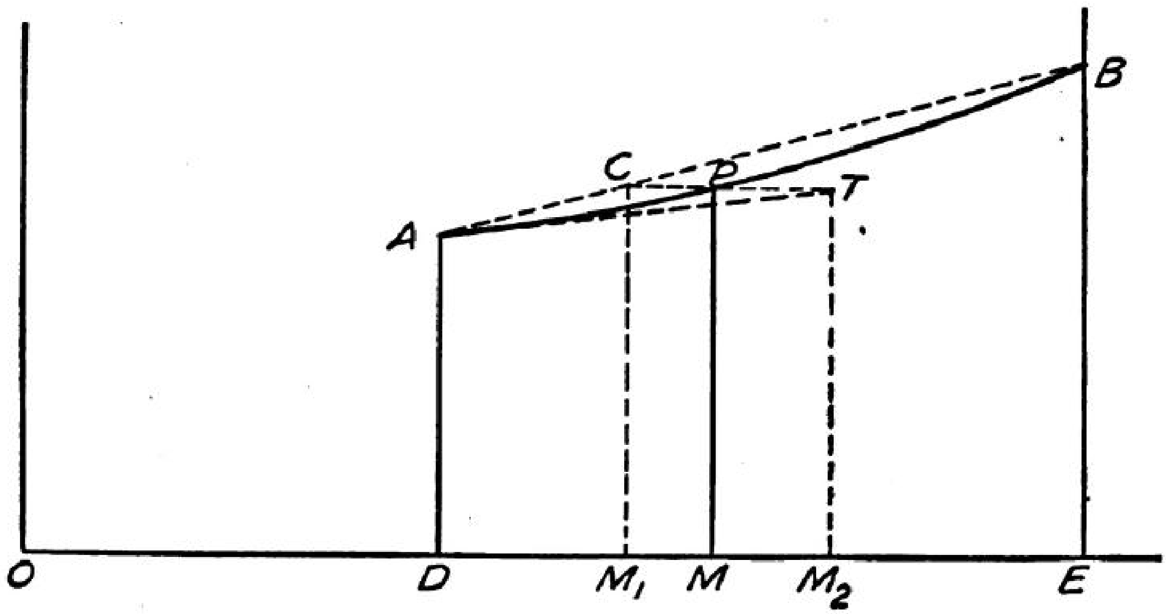
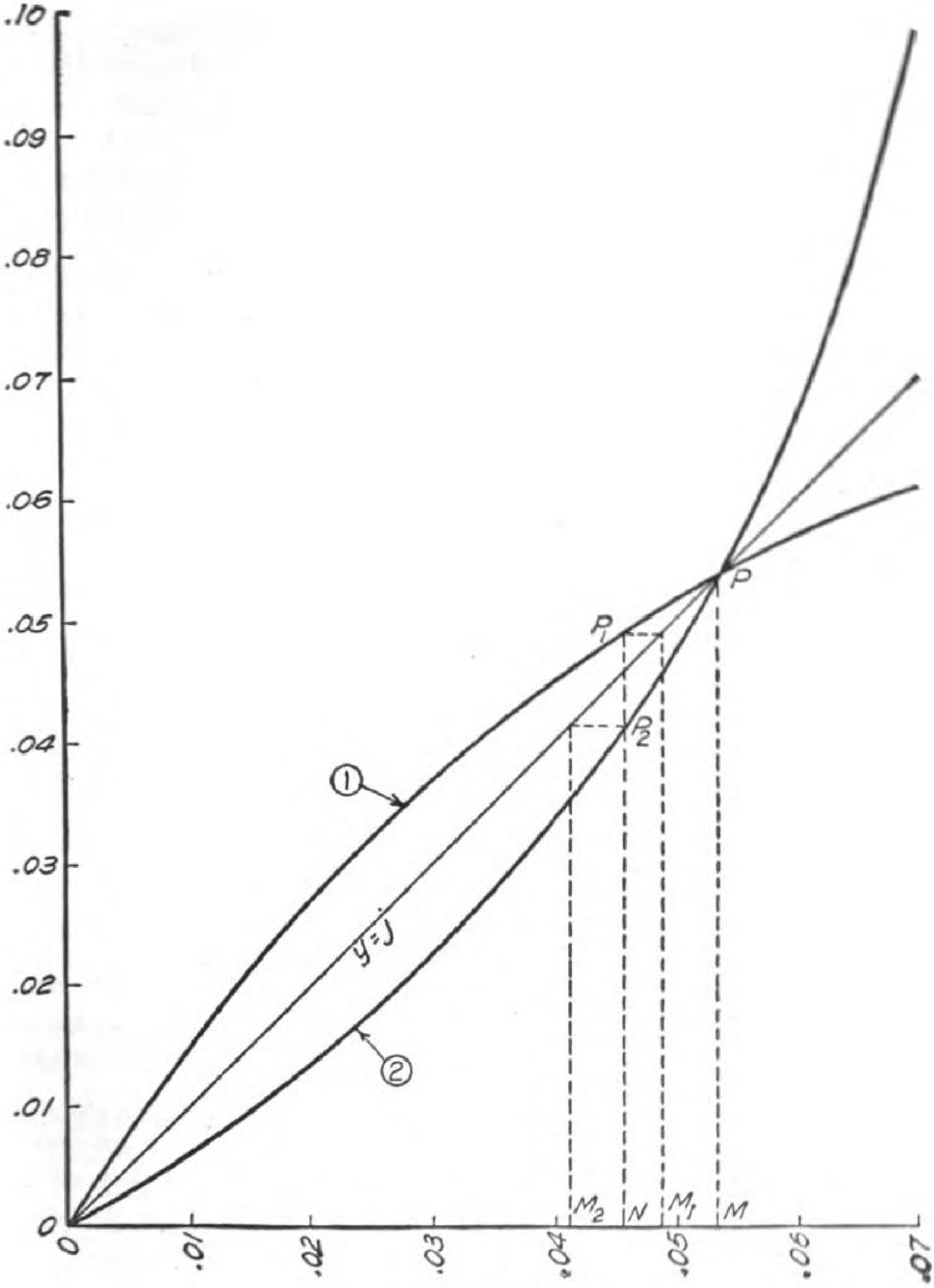


FIGURE 7.

To solve the equation $f(x) = k$ is to find the value of OM. Since $OM = a + DM$, this is equivalent to finding the value of DM. In the interpolation method the arc APB is replaced by the chord AB, the given ordinate, MP, takes the position M_1C, and DM_1 is found instead of DM. In Newton's method, if a is the approximation used, the arc APB is replaced by its tangent at A, the given ordinate, takes the position M_2T, and DM_2 is found instead of DM. When the arc APB is nearly straight both methods give excellent approximations.

The solution by the method of iteration of a rate equation in j is based on writing the equation in a form $j = f(j)$, where $f(j)$ is a



.10
.09
.08
.07
.06
.05
.04
.03
.02
.01
0
0
.01
.02
.03
.04
.05
.06
.07
P
P_1
P_2
1
2
y=j
M_2
N
M_1
M

FIGURE 8.

function of j whose value, $j'' = f(j')$, for a given approximate value j' of j is a closer approximation to the root than j'; likewise $f(j'')$ is a closer approximation than j''. By repetition or iteration of the process of computing successive values of $f(j)$ the root can be found to the accuracy desired. The rate equations which arise in elementary finance can usually be written in the form $j = f(j)$ in more than one way. For example, the equation $12102 = 1000\dfrac{1-(1+j)^{-20}}{j}$ can be solved readily for either j which occurs in it, and hence it may be written in the forms: $j = \dfrac{1-(1+j)^{-20}}{12.102}$; $j = (1 - 12.102\,j)^{-\frac{1}{20}} - 1$. The first of these forms is suited to the method of iteration, but the second is not. This may be seen by observing the graphs of

$$(1)\ y = \frac{1-(1+j)^{-20}}{12.102} \text{ and } (2)\ y = (1 - 12.102\,j)^{-\frac{1}{20}} - 1.$$

To solve either of the two forms of this rate equation is to find the abscissa OM of the point P in which either graph (1) or graph (2), Figure 8, is cut by the line $y = j$. From graph (1) it is seen that the first form of the j equation replaces any approximation ON by NP, $= OM_1$, which is a closer approximation to OM than ON; from graph (2) it is seen that the second form of the j equation replaces the approximation ON by $NP_2 = OM_2$ which is not as good an approximation to OM as ON. Observation of the graph of a like form of any rate equation will show if the form is suited to the method of iteration.

Some examples will now be solved to illustrate these methods.

EXAMPLE 1. An annuity of \$100 per year has a value of \$2065 at the end of 15 years at the interest rate i converted annually. Find the interest rate i.

Substituting into formula (1) and dividing both members by 100 gives, for the equation to be solved,

$$\frac{(1+i)^{15}-1}{i} = 20.65 \quad \text{or} \quad s_{\overline{15}|i} = 20.65$$

SOLUTION BY INTERPOLATION. In this case $s_{\overline{15}|i} = \dfrac{(1+i)^{15}-1}{i}$ can be used to construct a table of values for interpolation. By Table ~~IV~~ V $s_{\overline{15}|.04} = 20.02358764$ and $s_{\overline{15}|.045} = 20.78405429$. Hence i lies between .04

and .045. Interpolation from the table

i	$s_{\overline{15}\vert i}$
.04	20.0236
i	20.65
.045	20.7841

gives $\dfrac{i - .04}{.045 - .040} = \dfrac{.6264}{.7605}$.

Solving, $i = .0441$

Figure (6) in Art. 24 shows that the graph of $s_{\overline{15}|i}$ is similar to that in Figure (7). It follows that the root to be found is greater than .0441. By use of a 7-place table of logarithms it is found that $s_{\overline{15}|.0442} = 20.660$. Hence the root lies between .0441 and .0442, so that .0441 is correct to four decimals. Computing $s_{\overline{15}|.0441}$ and interpolating from the table

i	$s_{\overline{15}\vert i}$
.0441	20.644
i	20.65
.0442	20.660

gives $\dfrac{i - .0441}{.0442 - .0441} = \dfrac{6}{16}$.

Solving, $i = .04413$

It follows again that the root is greater than .04413. This is seen also from the fact that $s_{\overline{15}|.04413} = 20.649$. The root is less than .04414 since $s_{\overline{15}|.04414} = 20.650+$. Hence the root lies between .04413 and .04414.

The separate tables showing pairs of values of $s_{\overline{15}|i}$ and i which satisfy the relation $s_{\overline{15}|i} = \dfrac{(1+i)^{15} - 1}{i}$ can be combined into the single table

i	$s_{\overline{15}\vert i}$
.04	20.0236
.0441	20.644
.04413	20.649
i	20.650
.04414	20.650+
.0442	20.660
.045	20.7841

Finding the first five digits of the root is equivalent to constructing this table of pairs of corresponding values of $s_{\overline{15}|i}$ and i. The numbers in the first and the last rows are the first pair, those in the second and the next to the last are the second pair, and so on. A similar table can be constructed for any rate equation.

Solution by Newton's Method. The equation to be solved may be written

$$(1+i)^{15} - 20.65\,i - 1 = 0$$

One interpolation shows that the root is .0441 approximately. Replacing i by $.0441 + h$, expanding by the binomial theorem, and dropping powers of h higher than the first gives

$$(1.0441)^{15} - 1 - 20.65(.0441) + \left(15(1.0441)^{14} - 20.65\right) h = 0$$

On computing by seven-place logarithms, this gives

$$h = .000034 \text{ or } .000035$$
$$i = .044134 \text{ or } .044135$$

A second application of Newton's method, using $i = .04413 + h$, shows that $h = .000004$ or $.000005$. A seven-place table of logarithms is not of sufficient extent to show which of these values of h is correct.

EXAMPLE 2. The present value of an annuity of $1000 per year for twenty years is $12102. Find the interest rate i if interest is converted annually.

Substituting into formula (2) and dividing both members by 1000 gives for the equation to be solved,

$$\frac{1-(1+i)^{-20}}{i} = 12.102 \quad \text{or} \quad a_{\overline{20}|i} = 12.102.$$

SOLUTION BY INTERPOLATION. In this case $a_{\overline{20}|i} = \dfrac{1-(1+i)^{-20}}{i}$ can be used to construct a table of values for interpolation. By means of Table VI and a seven-place table of logarithms, the following table of pairs of values of $a_{\overline{20}|i}$ and i can be readily computed:

i	$a_{\overline{20}\|i}$
.05	12.4622
.0534	12.1106
i	12.1020
.0535	12.1005
.0550	11.9504

This table shows that the root lies between .0534 and .0535. Another interpolation gives $i = .05349$.

SOLUTION BY ITERATION.* As seen by the solution by interpolation the root i lies between .05 and .055. The equation to be solved can be written in the following form, which, by the above discussion, is suited to the method of iteration:

$$i = \frac{1-(1+i)^{-20}}{12.102}$$

Employing this form and using arrows to indicate substitutions, the work can be outlined as follows:

$$
\begin{array}{lcl}
.05 \longrightarrow .0515 & & .055 \longrightarrow .0534 \\
 & \text{Average} = .0529 & \\
.053 \longrightarrow .0532 & & .054 \longrightarrow .0538 \\
 & \text{Average} = .0535 & \\
.0535 \longrightarrow .05349 & &
\end{array}
$$

EXERCISES

1. An annuity having $R = 10$, $n = 5$, $r = \frac{1}{2}$ has an amount of $112.30. Find $\frac{j}{2}$ correct to four places of decimals if $m = 2$. Ans. .0255.

2. An annuity having $R = 100$, $n = 5$, $r = \frac{1}{4}$ has a present value of $1700. Find $\frac{j}{4}$ correct to four decimals if $m = 4$.

* See paper by Forsythe referred to on page 69.

3. An electric washer was priced at \$100 cash or \$5 cash and \$8.50 at the end of each month for twelve months. If $m = 12$, find $\frac{j}{12}$ correct to five decimals, which will make the two prices equivalent. Ans. .01111.

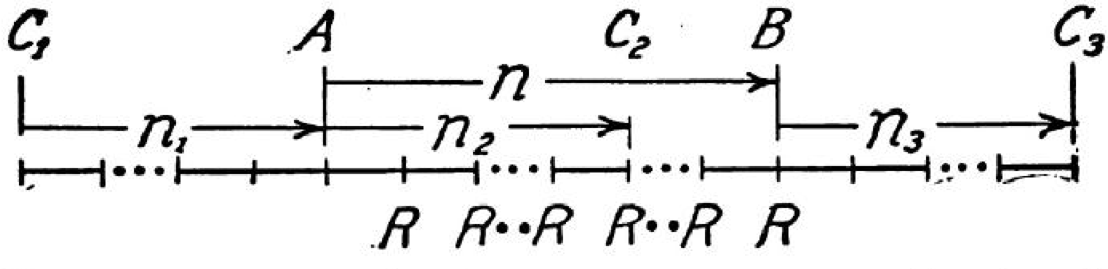
27. The value of an annuity at any time. Deferred and forborne annuities. An annuity whose term begins n_1 years after a specified time is called a *deferred annuity* with respect to this time; it is said to be deferred n_1 years. An annuity whose term begins n_1 years before a specified time is called a *forborne annuity;* it is said to be forborne n_1 years. Formulas (1), (3) and (2), (4) give the values of an annuity at the end and at the beginning of its term.

The expression for the value of a deferred or a forborne annuity can be written at once by use of either one of the following methods:

Method 1. To find the value of an annuity at any time, apply Theorem I, Art. 15, Chapter I to V_o or V_n.

Method 2. To find the value of an annuity at any time, resolve the annuity into the difference or sum of two annuities each of which has the beginning or end of its term at the given time, and then find the difference or sum of the values of these component annuities at this time.

The following line diagram can be used to visualize these two methods:



AB represents an annuity of rent R, term n years, and rent period r years. The arrows indicate increasing time. The value of this annuity at C_1 will be denoted by V_{-n_1}, at A by V_o, at C_2 by V_{n_2}, at B by V_n, and at C_3 by $V_{(n+n_3)}$.

By Method 1, the value at C_1 of the annuity AB deferred n_1 years is given by

$$V_{-n_1} = V_o\left(1 + \frac{j}{m}\right)^{-mn_1}$$

By Method 2, the value at C_1 of the annuity AB deferred n_1 years can be found by first filling in the missing values of R between C_1 and A and then subtracting the present value of the annuity C_1A from that of the annuity C_1B. This gives, using formula (4),

$$V_{-n_1} = R\frac{a_{\overline{m(n+n_1)}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}} - R\frac{a_{\overline{mn_1}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}} = R\frac{a_{\overline{m(n+n_1)}|\frac{j}{m}} - a_{\overline{mn_1}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}}$$

Analogous expressions for the values of the annuity AB at C_2 and C_3 are given in Exercises 3 and 4 at the end of this article.

Some examples will now be solved to illustrate these methods.

Example 1. Find the present value of an annuity of \$100 every six months for five years, deferred three years, at ($j = .055$, $m = 2$).

Solution by method 1.

$$\begin{aligned} V_{-3} &= V_0\,(1.0275)^{-6} \\ &= 100\;a_{\overline{10}|.0275}\,(1.0275)^{-6} \\ &= \$734.22 \end{aligned}$$

Solution by method 2.

$$\begin{aligned} V_{-3} &= 100(a_{\overline{16}|.0275} - a_{\overline{6}|.0275}) \\ &= \$734.22 \end{aligned}$$

Example 2. Find the value of an annuity of \$100 every six months for five years, forborne eight years, that is, three years after the payments are discontinued, at ($j = .06$, $m = 2$).

Solution by method 1.

$$\begin{aligned} V_8 &= V_5(1.03)^6 \\ &= 100\;s_{\overline{10}|.03}\,(1.03)^6 \\ &= \$1368.85 \end{aligned}$$

Solution by method 2.

$$\begin{aligned} V_8 &= 100(s_{\overline{16}|.03} - s_{\overline{6}|.03}) \\ &= \$1368.85 \end{aligned}$$

Example 3. Find the value of the annuity in Example 1 if it is forborne two years, that is, find its value at the time of the fourth payment.

Solution by method 1.

$$\begin{aligned} V_2 &= V_0(1.0275)^4 \\ &= 100\;a_{\overline{10}|.0275}\cdot(1.0275)^4 \\ &= \$963.04 \end{aligned}$$

Solution by method 2.

$$V_2 = 100(s_{\overline{4}|.0275} + a_{\overline{6}|.0275})$$
$$= \$963.04$$

Exercises 1–4 which follow refer to the annuity AB diagrammed above.

Exercise 1. Show that

$$V_{-n_1} = R\frac{\left(1+\frac{j}{m}\right)^{-mn_1} - \left(1+\frac{j}{m}\right)^{-m(n+n_1)}}{\left(1+\frac{j}{m}\right)^{mr} - 1} \quad \text{By Method 1.}$$

$$= R\frac{\dfrac{1-\left(1+\frac{j}{m}\right)^{-m(n+n_1)}}{\frac{j}{m}} - \dfrac{1-\left(1+\frac{j}{m}\right)^{-mn_1}}{\frac{j}{m}}}{\dfrac{\left(1+\frac{j}{m}\right)^{mr}-1}{\frac{j}{m}}}$$

$$= R\frac{a_{\overline{m(n+n_1)}|\frac{j}{m}} - a_{\overline{mn_1}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}}$$

This result shows that the two methods give equivalent results in case of a deferred annuity.

Exercise 2. When $R = r = 1$, V_{-n_1} is denoted by $_{n_1|}a_{\overline{n}|i}$; when $R = r = \dfrac{1}{p}$, V_{-n_1} is denoted by $_{n_1|}a^{(p)}_{\overline{n}|i}$. Show by use of the result of Exercise 1, that

$$_{n_1|}a_{\overline{n}|i} = a_{\overline{n+n_1}|i} - a_{\overline{n_1}|i}$$
$$_{n_1|}a^{(p)}_{\overline{n}|i} = a^{(p)}_{\overline{n+n_1}|i} - a^{(p)}_{\overline{n_1}|i}$$

Exercise 3. Show that, $(n_2 < n)$

$$V_{n_2} = R\frac{\left(1+\frac{j}{m}\right)^{mn_2} - \left(1+\frac{j}{m}\right)^{m(n_2-n)}}{\left(1+\frac{j}{m}\right)^{mr} - 1}$$

$$= R\frac{s_{\overline{mn_2}|\frac{j}{m}} + a_{\overline{m(n-n_2)}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}}$$

EXERCISE 4. Show that

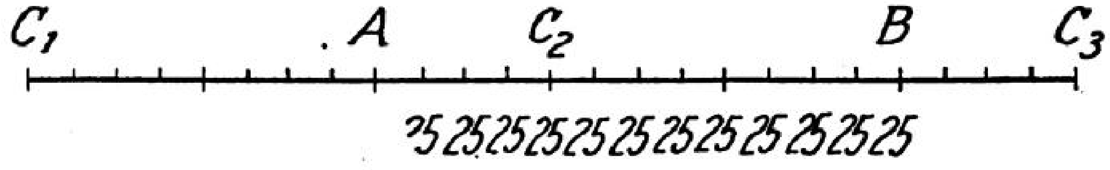
$$V_{(n+n_2)} = R\,\frac{\left(1+\frac{j}{m}\right)^{m(n+n_2)} - \left(1+\frac{j}{m}\right)^{mn_2}}{\left(1+\frac{j}{m}\right)^{mr} - 1}$$

$$= R\,\frac{s_{\overline{m(n+n_2)}|\frac{j}{m}} - s_{\overline{mn_2}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}}$$

EXERCISE 5. Use the results of Exercises 1, 3, and 4 to show that Methods 1 and 2 give the same result for the value of an annuity at any time.

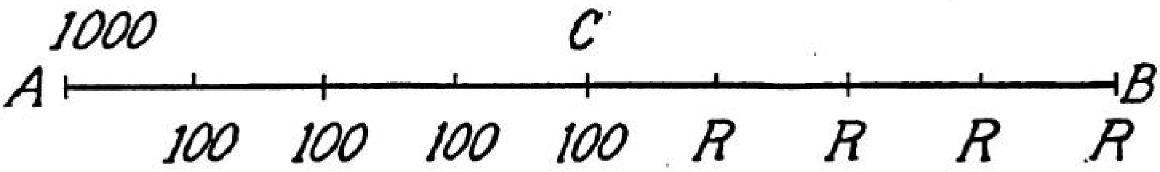
EXERCISES

1. Each section of the following diagram represents three months:



Find the values of the annuity AB at C_1, at C_2, at C_3, (a) at the rate ($j = .045$, $m = 4$); (b) at the rate ($j = .06$, $m = 12$); (c) at the rate ($j = .045$, $m = 2$). How long is the annuity AB deferred with reference to C_1? How long is it forborne with reference to C_2, and how long with reference to C_3?

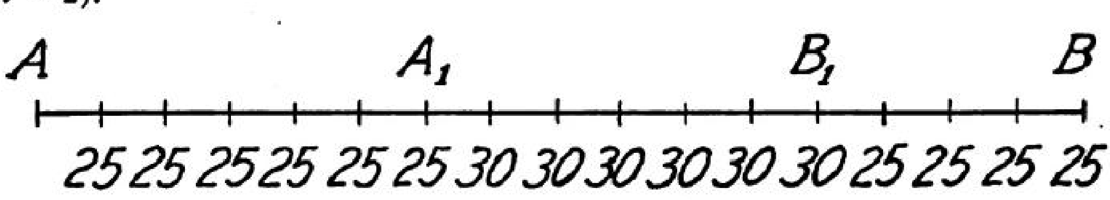
2. Each section of the following diagram represents six months:



The set of sums below the line is equivalent to that above it. Find R by equating the values of these sets at A; at C; at B. Use ($j = .05$, $m = 2$).

Ans. \$183.03.

3. Find the values at A and at B of the set of sums, represented by the diagram below, where each section of the line represents one year, at ($j = .06$, $m = 2$).

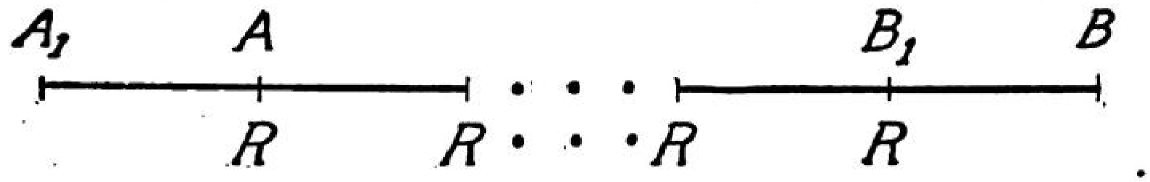


[Resolve the set of sums into the annuity AB whose rent is \$25 and the annuity A_1B_1 whose rent is \$5]. Ans. \$268.29; \$690.87.

4. A man purchased a house for \$6800, paying \$1200 cash, and agreeing to make monthly payments on the balance. He paid \$50 for 65 months, \$40 for 15 months, and \$55 for 20 months. What was the amount of the debt after the last payment if ($j = .06$, $m = 12$)? How many additional \$55 payments will reduce the debt to less than \$55? If the debt is paid in full one month after it is reduced to less than \$55, find the last payment.

Ans. \$2821.43; 59; \$23.33.

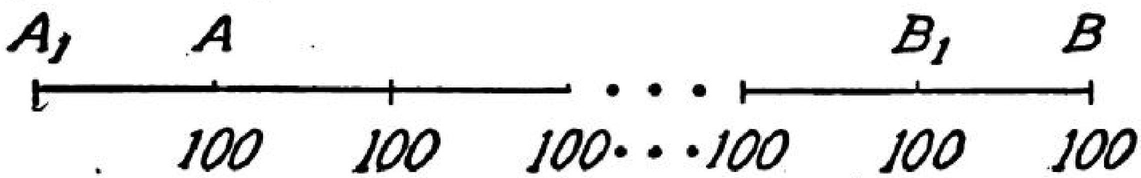
28. The value of an annuity due at any time. In an annuity due the rent is paid at the beginning of each rent period; in an annuity it is paid at the end of each rent period. It follows that an annuity due of given rent, rent period, and term, is equivalent to an annuity of the same rent, the same rent period, but whose term begins and ends one rent period earlier. A line diagram may be used to represent the annuity due and its equivalent annuity:



In this diagram AB represents an annuity due whose term begins at A and ends at B, and A_1B_1 represents the equivalent annuity whose term begins at A_1 and ends at B_1. The value at a given time of any annuity due, AB, is then the value at this time of the annuity A_1B_1. It follows that the methods in Art. 27 are directly applicable to the problem of finding the value at a given time of an annuity due.

Example 1. Find the present value of an annuity due having $R = 100$, $n = 20$, $r = \frac{1}{2}$ at ($j = .06$, $m = 2$).

In the following diagram * let AB represent this annuity due and A_1B_1 the equivalent annuity.



The present value of the annuity due, AB, is then the value at A of the annuity A_1B_1.

Solution by method 1.

$$
\begin{aligned}
V_{\frac{1}{2}} &= V_o(1.03) \\
&= 100\, a_{\overline{40}|.03}\,(1.03) \\
&= \$2380.82
\end{aligned}
$$

* Delete the 100 under B.

SOLUTION BY METHOD 2.

$$\begin{aligned} V_{\frac{1}{2}} &= 100 + 100\, a_{\overline{39}|.03} \\ &= \$2380.82 \end{aligned}$$

EXAMPLE 2. Find the value of the annuity due in Example 1, if it is deferred 10 years; if it is forborne 30 years.

SOLUTION BY METHOD 1.

$$\begin{aligned} V_{9\frac{1}{2}} &= V_o\,(1.03)^{-19} \\ &= 100\, a_{\overline{40}|.03}\,(1.03)^{-19} \\ &= \$1318.20 \end{aligned} \qquad \begin{aligned} V_{30\frac{1}{2}} &= V_o\,(1.03)^{61} \\ &= 100\, a_{\overline{40}|.03}\,(1.03)^{61} \\ &= \$14026.86 \end{aligned}$$

SOLUTION BY METHOD 2.

$$\begin{aligned} V_{9\frac{1}{2}} &= 100 + 100(a_{\overline{39}|.03} - a_{\overline{19}|.03}) \\ &= \$1318.20 \end{aligned} \qquad \begin{aligned} V_{30\frac{1}{2}} &= 100(s_{\overline{81}|.03} - s_{\overline{21}|.03}) \\ &= \$14026.86 \end{aligned}$$

The symbols used for the value of an annuity due are the same as those for an annuity with the exception that a is replaced by $\mathbf{a}$ and s by $\mathbf{s}$. For example, $\mathbf{a}_{\overline{n}|}$ represents the present value, and $\mathbf{s}_{\overline{n}|}$ the amount of an annuity due of rent 1 payable annually for n years.

EXERCISE 1. Show that

$$\begin{aligned} \mathbf{a}_{\overline{n}|i} &= 1 + a_{\overline{n-1}|i} = a_{\overline{n}|i}\,(1+i) \\ \mathbf{s}_{\overline{n}|i} &= s_{\overline{n+1}|i} - 1 = s_{\overline{n}|i}\,(1+i) \end{aligned}$$

[Use methods 1 and 2, Art. 27.]

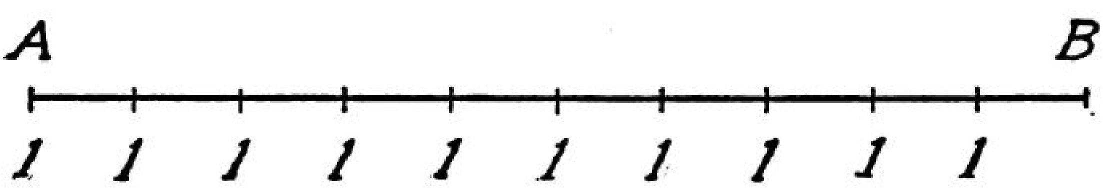
EXERCISE 2. Show that

$$\mathbf{a}^{(p)}_{\overline{n}|i} = \frac{1}{p} + a^{(p)}_{\overline{n-\frac{1}{p}}|i} = a^{(p)}_{\overline{n}|i} \cdot (1+i)^{\frac{1}{p}}$$

$$\mathbf{s}^{(p)}_{\overline{n}|i} = s^{(p)}_{\overline{n+\frac{1}{p}}|i} - \frac{1}{p} = s^{(p)}_{\overline{n}|i} \cdot (1+i)^{\frac{1}{p}}$$

EXERCISES

1. Each section of the diagram represents one year:



Find the value of the annuity due AB at A, and at B, if ($j = .06$, $m = 1$).

2. Show that the annuity due AB of annual rent 1 in Exercise 1 is equivalent to the annuity AB of annual rent 1.06 if ($j = .06$, $m = 1$). Use this fact to show that

$$\mathbf{s}_{\overline{10}|.06} = (1.06)\, s_{\overline{10}|.06}$$
$$\mathbf{a}_{\overline{10}|.06} = (1.06)\, a_{\overline{10}|.06}$$

3. Show that an annuity due of rent R is equivalent to an annuity of rent $R\left(1+\frac{j}{m}\right)^{mr}$, having the same term and rent period if the interest rate j is converted m times a year.

4. What is the annual rent payable in advance which is equivalent to a monthly rental of \$60 payable at the beginning of each month if ($j = .06$, $m = 1$)? (See Exercise 7, Art. 25.) Ans. \$701.12.

5. Deposits of \$100 were made in a savings bank at the beginning of each quarter year for 5 years. Find the amount of these deposits at the end of 5 years; at the end of 10 years; at the time of the last deposit. Use ($j = .04$, $m = 4$). Ans. \$2223.92; \$2713.60; \$2201.90.

6. Same as Exercise 5, except that the deposits were made at the beginning of each month. Ans. \$6649.70; \$8213.89; \$6627.68.

29. The value of a perpetuity. By the definition in Art. 1, a perpetuity is an annuity whose term becomes infinite. From this definition it follows that the value of a perpetuity at any time can be found by making n infinite in the expression for the value of an annuity at the given time. The result of making n infinite in any annuity formula can be written at once by noting that $\underset{k\to\infty}{L}\left(1+\frac{j}{m}\right)^{-k} = 0$ and $\underset{k\to\infty}{L}\left(1+\frac{j}{m}\right)^{k} \to \infty$ and hence that $a_{\overline{\infty}|\frac{j}{m}} = \dfrac{1}{\frac{j}{m}}$ and $s_{\overline{\infty}|\frac{j}{m}} \to \infty$.

$$\text{For example, when } n \to \infty,\ V_o = R\frac{1}{\left(1+\frac{j}{m}\right)^{mr} - 1} \tag{2_3}$$

$$= R\frac{1}{\frac{j}{m}s_{\overline{mr}|\frac{j}{m}}} \tag{4_3}$$

The value of a perpetuity at any time other than the beginning of its term can be found by Methods 1 and 2, Art. 11. In applying Method 2, the given perpetuity is resolved into the sum or difference of a perpetuity and an annuity.

Exercise 1. Show that, when $n \to \infty$, the value of a perpetuity deferred n_1 years is given by

$$V_{-n_1} = R\frac{\left(1+\frac{j}{m}\right)^{-mn_1}}{\left(1+\frac{j}{m}\right)^{mr}-1} = R\frac{\dfrac{\frac{1}{j}-a_{\overline{mn_1}|\frac{j}{m}}}{m}}{s_{\overline{mr}|\frac{j}{m}}}$$

Exercise 2. Show that, when $n \to \infty$, the value of a perpetuity forborne n_1 years is given by

$$V_{n_1} = R\frac{\left(1+\frac{j}{m}\right)^{mn_1}}{\left(1+\frac{j}{m}\right)^{mr}-1} = R\frac{\dfrac{s_{\overline{mn_1}|\frac{j}{m}}+\frac{1}{j}}{m}}{s_{\overline{mr}|\frac{j}{m}}}$$

Example 1. Find the present value of the perpetuity ($R = 100$, $n \to \infty$, $r = 5$) at the rate ($j = .06$, $m = 2$).

Solution. By formula (4_3)

$$V_o = 100\frac{1}{.03\, s_{\overline{10}|.03}}$$
$$= \frac{100}{.03}(.08723051) = \$290.77$$

Exercise. Solve Example 1 by using $V_o = \dfrac{100}{(1.03)^{10}-1}$ [Formula (2_3)]

Example 2. Find the present value of the perpetuity ($R = 100$, $n \to \infty$, $r = \frac{1}{2}$) at the rate ($j = .06$, $m = 4$).

Solution. By formula (4_3)

$$V_o = \frac{100}{.015}\frac{1}{s_{\overline{2}|.015}}$$
$$= \frac{100}{.015}(.49627792) = \$330.85$$

EXERCISES

1. Find the present value of an annuity having $R = r = 1$ at ($j = .06$, $m = 1$) if $n = 10$; if $n = 100$; if $n = 150$; if $n \to \infty$.
Ans. \$7.36; \$16.62; \$16.66; \$16.67.

2. Find the present value of a perpetuity whose term is deferred 10 years if $R = 10$ and $r = \frac{1}{4}$ at ($j = .06$, $m = 4$); find the value if the term is forborne 10 years. Ans. \$737.76; \$2427.73.

3. How much could a railroad company afford to spend to eliminate a dangerous crossing requiring the attention of two watchmen at \$150 a month each? Use ($j = .05$, $m = 2$). Ans. \$72746.32.

4. If ($j = .05$, $m = 2$) what is the amount of an endowment which will provide a perpetuity of $1000 at the beginning of each year? Ans. $20753.09.

5. By the terms of a will an annual perpetuity of $1000 is equally divided between two charities. If the first charity receives the entire income until it has received its share, find the number of payments it receives and the amount of the last payment if ($j = .05$, $m = 2$). Ans. 29; $35.48.

6. Derive formula (2_3) by use of the formula for the sum of a geometric progression when the number of terms become infinite.

7. Derive formula (2_3) by noting that V_o is the principal of an investment which yields R dollars of interest at the end of each r years.

[In this type of investment, the principal is not returned.]

8. Solve Example 1 above by use of the principle stated in Exercise 7.

30. The value of a continuous annuity. A continuous annuity is one having an infinite number of rent payments each year and a fixed annual rent K; that is, in a continuous annuity $R = \frac{K}{p}$, and p becomes infinite. In Art. 14, Chapter I, it was seen, by equating two expressions for the force of interest, that $L_{x\to\infty} x[(1 + i)^{\frac{1}{x}} - 1] = \log_e(1 + i)$. Upon replacing x by $\frac{p}{m}$ and i by $\frac{j}{m}$ and multiplying by m it follows from this limit that

$$L_{p\to\infty} p\left[\left(1 + \frac{j}{m}\right)^{\frac{m}{p}} - 1\right] = m \log_e\left(1 + \frac{j}{m}\right)$$

and hence that

$$L_{p\to\infty}\left(p s_{\overline{\frac{m}{p}}|\frac{j}{m}}\right) = \frac{m^2}{j}\log_e\left(1 + \frac{j}{m}\right).$$

By means of these results the value of a continuous annuity at any time can be written from the formula for the value of an annuity at that time. For example,

$$\text{When } p\to\infty \text{ and } R = \frac{K}{p},\ V_o = K\frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{m \log_e\left(1 + \frac{j}{m}\right)} \qquad (2_4)$$

$$= K\frac{j}{m^2}\frac{a_{\overline{mn}|\frac{j}{m}}}{\log_e\left(1 + \frac{j}{m}\right)} \qquad (4_4)$$

When $p \rightarrow \infty$ and $R = \frac{K}{p}$, $V_n = K \dfrac{\left(1+\frac{j}{m}\right)^{mn} - 1}{m \log_e\left(1+\frac{j}{m}\right)}$ (1_4)

$$= K \frac{j}{m^2} \frac{s_{\overline{mn}|\frac{j}{m}}}{\log_e\left(1+\frac{j}{m}\right)} \qquad (3_3)$$

When $K = m = 1$ and $p \rightarrow \infty$, V_o is denoted by $\bar{a}_{\overline{n}|i}$, and V_n is denoted by $\bar{s}_{\overline{n}|i}$.

Exercise. Show that

$$\bar{a}_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{\log_e(1+i)} = \frac{i}{\delta} a_{\overline{n}|i}$$

$$\bar{s}_{\overline{n}|i} = \frac{(1+i)^n - 1}{\log_e(1+i)} = \frac{i}{\delta} s_{\overline{n}|i}$$

Example. Find the present value of the continuous annuity ($K = 100$, $n = 10$) at the rate ($j = .06$, $m = 2$).

Solution. By formula (4_4)

$$V_o = 100(.015) \frac{a_{\overline{20}|.03}}{\log_e(1.03)}$$

$$= \$754.99$$

EXERCISES

1. At ($j = .06$, $m = 1$) find V_n for each of the following annuities: $K = Rp = 1000$, $n = 10$, $\frac{1}{r} = p = 2, 12, 52, 365$ and for $\frac{1}{r} = p \rightarrow \infty$.

Ans. \$13375.62; \$13539.46; \$13564.82; \$13571.60; \$13572.38.

2. Compute the values of $1000\, a^{(365)}_{\overline{10}|.06}$ and $1000\, \bar{a}_{\overline{10}|.06}$

Ans. \$7578.31; \$7578.75.

31. An extension of the compound interest formula. The interest payments on a principal, P, bearing the interest rate j payable m times per year form an annuity whose rent is $\frac{Pj}{m}$ and whose rent period is $\frac{1}{m}$. By formula (3) the amount of this annuity at the rate j converted m times per year is given by $V_n = \frac{Pj}{m} s_{\overline{mn}|\frac{j}{m}}$. It follows that the compound amount of P for n

years at the rate j converted m times per year can be written in the form, $S = P\left(1 + \frac{j}{m}\, s_{\overline{mn}|\frac{j}{m}}\right)$. The two formulas

$$S = P\left(1 + \frac{j}{m}\right)^{mn}$$

$$S = P\left(1 + \frac{j}{m}\, s_{\overline{mn}|\frac{j}{m}}\right)$$

give the same value for S. In the derivation of each of these formulas, the interest payments bear the same rate as the original principal.

If often happens in business transactions that the interest payments on P are invested at rates which differ from the rate which P yields. In such cases the above formulas are not applicable. The method used in getting the second formula is applicable, however, to any case. Let P bear the rate j payable m times per year. When the interest payments bear the rate j' payable m times per year, the formula for S becomes

$$S = P\left(1 + \frac{j}{m}\, s_{\overline{mn}|\frac{j'}{m}}\right)$$

When the interest payments bear the rate j' payable m' times per year the formula for S becomes

$$S = P\left(1 + \frac{j}{m}\, \frac{s_{\overline{m'n}|\frac{j'}{m'}}}{s_{\overline{\frac{m'}{m}}|\frac{j'}{m'}}}\right)$$

Still more general formulas for S can be gotten by separating the interest payments on P into sets of one or more each, and then accumulating these sets at different rates by the use of the formulas for the amounts of a single sum, an annuity, and a forborne annuity. Many interesting formulas can be found in this way. In problems in elementary finance these formulas need not be used; it is better to make direct use of the method just stated, by means of which they can be written.

Each of these formulas, when solved for P, gives an analogous formula for the present value of a sum.

EXERCISE 1. If $S = P\left(1 + \frac{j}{m} s_{\overline{mn}|\frac{j}{m}}\right)$ show that

$$P = \frac{S}{\left(1 + \frac{j}{m} s_{\overline{mn}|\frac{j}{m}}\right)} = S\left(1 - \frac{j}{m} a_{\overline{mn}|\frac{j}{m}}\right).$$

EXERCISE 2. If $S = P(1 + i\, s_{\overline{n}|i'})$ show that

$$P = \frac{S}{(1 + i\, s_{\overline{n}|i'})} \neq S(1 - i\, a_{\overline{n}|i'}).$$

EXAMPLE. A \$1000 bond bearing 4% payable semi-annually and maturing in 5 years at par is bought at its face value. Find the amount of this investment at the end of the five years if the buyer can invest the interest payments (1) at ($j = .04, m = 2$); (2) at ($j = .05, m = 2$); (3) at ($j = .06, m = 4$); (4) at ($j = .05, m = 2$) during the first three years and at ($j = .06, m = 2$) during the last two years.

SOLUTION. In this example the interest payments on the bond form the annuity ($R = 20, n = 5, r = \frac{1}{2}$). The amount of this annuity

at ($j = .04, m = 2$) is $V_5 = 20\, s_{\overline{10}|.02} = 218.99$;

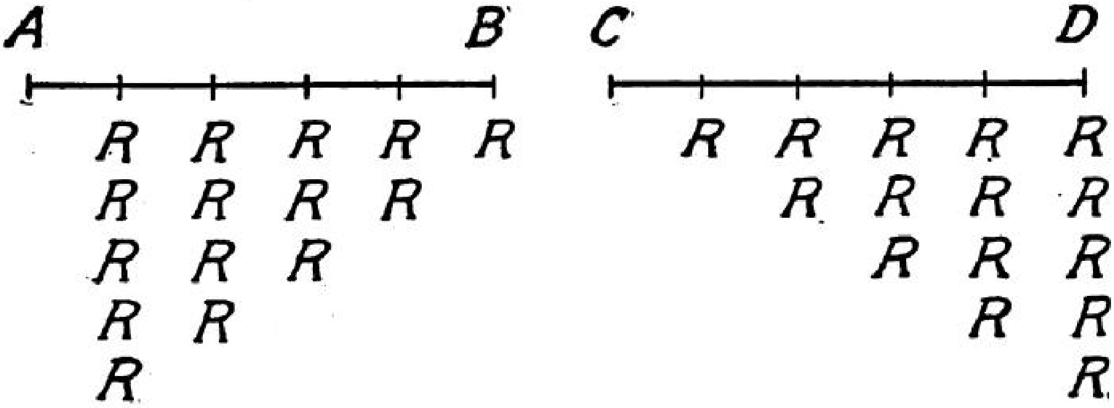
at ($j = .05, m = 2$) is $V_5 = 20\, s_{\overline{10}|.025} = 224.07$;

at ($j = .06, m = 4$) is $V_5 = 20\, s_{\overline{20}|.015} = 229.52$;

at $\begin{pmatrix} j = .05, m = 2 \text{ for 3 yrs.} \\ j = .06, m = 2 \text{ for 2 yrs.}\end{pmatrix}$ is $V_5 = 20(s_{\overline{10}|.025} - s_{\overline{4}|.025}) + 20\, s_{\overline{4}|.03} = 224.69.$

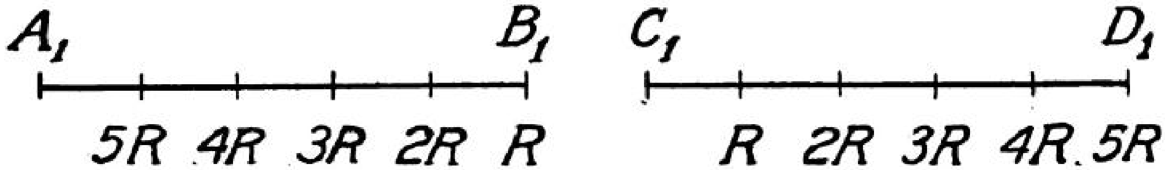
Hence the amounts of the investments are (1) \$1218.99, (2) \$1224.07, (3) \$1229.52, (4) \$1224.69.

32. The value of a set of annuities. Increasing and decreasing annuities. Some transactions in finance involve sets of annuities. The value at any given time of a set of annuities can be found by adding the values at that time of the separate annuities which compose the set. The values of some sets can be found more easily, however, by basing the computations on simplified forms



of the expressions for the sum of the values of the given sets. The line diagrams on page 86, each section of which denotes r years, represent such sets.

The terms of the sets in AB all begin at A; the terms of those in CD all end at D. In each case the terms of the separate annuities are $r, 2r, 3r, 4r, 5r$. In general sets of these types, $5r$ is replaced by n. The separate sets represented by each of these diagrams can be combined into a single set by adding vertically. These single sets can be represented by the diagrams



When $5r$ is replaced by n the set represented by A_1B_1 is called a *decreasing annuity* of term n, and that represented by C_1D_1 is called an *increasing annuity* of term n.

In this article simple formulas are found for the values of increasing and decreasing annuities of the kinds illustrated by the above diagrams. Symbols for the values of these increasing and decreasing annuities may be obtained by prefixing the letters I and D to the symbols used in the preceding articles. For example, $(Is_{\overline{n}|i})$ denotes the amount at the annual interest rate i of an increasing annuity of term n and rent period 1, whose successive rent payments are $1, 2, \cdots, n$.

The expression for the present value of a decreasing annuity, obtained by writing the sum of the present values of the annuities which compose it, as illustrated by the diagrams A_1B_1 and AB, can be changed into a simple form. The process is as follows:

$$(DV_o) = R\left\{\frac{a_{\overline{mr}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}} + \frac{a_{\overline{2\,mr}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}} + \cdots + \frac{a_{\overline{mn}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}}\right\} \text{ by formula (4),}$$

$$= \frac{R}{s_{\overline{mr}|\frac{j}{m}}} \cdot \frac{\frac{n}{r} - \left[\left(1+\frac{j}{m}\right)^{-mr} + \left(1+\frac{j}{m}\right)^{-2mr} + \cdots + \left(1+\frac{j}{m}\right)^{-mn}\right]}{\frac{j}{m}}$$

by formula (2_1),

$$= \frac{R}{s_{\overline{mr}|\frac{j}{m}}} \frac{\frac{n}{r} - \frac{a_{\overline{mn}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}}}{\frac{j}{m}}$$

$$= R \frac{\frac{n}{r} - \frac{a_{\overline{mn}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}}}{\frac{j}{m} s_{\overline{mr}|\frac{j}{m}}} \tag{8}$$

When $mr = 1$ $\qquad (DV_o) = R \dfrac{mn - a_{\overline{mn}|\frac{j}{m}}}{\frac{j}{m}}$ $\qquad (8_1)$

When $m = 1, r = 1$ $\;(DV_o) = R \dfrac{n - a_{\overline{n}|i}}{i}$ $\qquad (8_2)$

Exercise. Find a formula for (DV_n), the amount of a decreasing annuity. [Use Theorem I and formula (8).]

The expression for the amount of an increasing annuity obtained by writing the sum of the amounts of the annuities which compose it, as illustrated by the diagrams C_1D_1 and CD, can be changed into a simple form. The process is analogous to that used in deriving formula (8).

$$(\mathrm{IV}_n) = R \left\{ \frac{s_{\overline{mr}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}} + \frac{s_{\overline{2mr}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}} + \cdots + \frac{s_{\overline{mn}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}} \right\} \text{ by formula (3)}$$

$$= \frac{R}{s_{\overline{mr}|\frac{j}{m}}} \frac{\left(1 + \frac{j}{m}\right)^{mr} + \left(1 + \frac{j}{m}\right)^{2mr} + \cdots + \left(1 + \frac{j}{m}\right)^{mn} - \frac{n}{r}}{\frac{j}{m}}$$

by formula (1_1)

$$= \frac{R}{s_{\overline{mr}|\frac{j}{m}}} \frac{\left(1 + \frac{j}{m}\right)^{mr}\left[1 + \left(1 + \frac{j}{m}\right)^{mr} + \cdots + \left(1 + \frac{j}{m}\right)^{m(n-r)}\right] - \frac{n}{r}}{\frac{j}{m}}$$

$$= \frac{R}{s_{\overline{mr}|\frac{j}{m}}} \frac{\left(1+\frac{j}{m}\right)^{mr} \dfrac{s_{\overline{mn}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}} - \dfrac{n}{r}}{\dfrac{j}{m}}$$

$$= R \frac{\left(1+\frac{j}{m}\right)^{mr} \dfrac{s_{\overline{mn}|\frac{j}{m}}}{s_{\overline{mr}|\frac{j}{m}}} - \dfrac{n}{r}}{\dfrac{j}{m} s_{\overline{mr}|\frac{j}{m}}} \tag{9}$$

When $mr = 1$, $$(IV_n) = R \frac{\left(1+\frac{j}{m}\right) s_{\overline{mn}|\frac{j}{m}} - mn}{\dfrac{j}{m}} \tag{9_1}$$

When $m = 1, r = 1,$ $$(IV_n) = R \frac{(1+i)s_{\overline{n}|i} - n}{i} \tag{9_2}$$

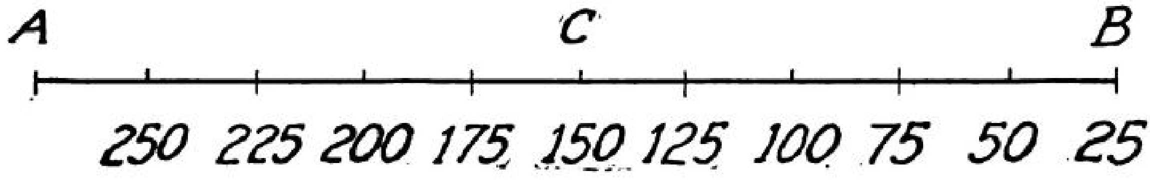
EXERCISE 1. Show that $(Is_{\overline{n}|i}) = \dfrac{s_{\overline{n+1}|i} - (n+1)}{i}$

EXERCISE 2. Find a formula for (IV_o). [Use Theorem I and formula (9).]

The processes used for finding the values of these increasing and decreasing annuities can be used for finding the values of other sets of annuities. (See Exercise 3, below.)

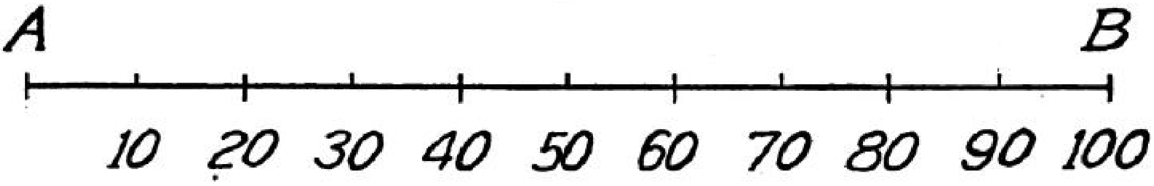
EXERCISES

1. Each section of the following diagram represents six months:



Find the value of the decreasing annuity AB at A; at B; and at C. Use ($j = .04, m = 2$). Ans. \$1271.77; \$1550.28; \$1376.60.

2. Each section represents six months:



Find the values of the increasing annuity AB at B and at A. Use ($j = .06$, $m = 2$). Ans. \$602.60; \$448.39.

3. Show that the present value at ($j = .04$, $m = 2$) of the set of annuities having $R = 25$, $r = \frac{1}{2}$, and $n = 5, 7, 9, 11, 13, 15, 17, 19, 21$, and 23 is

$$\frac{25}{.02}\left(10 - \frac{v^{10} - v^{50}}{1 - v^4}\right) \text{ where } v = (1.02)^{-1}$$

Compute the value of this expression. Ans. \$5133.07.

4. A bank established a Christmas savings club which required that each member deposit 5 cents at the end of the first week, 10 cents at the end of the second, 15 cents at the end of the third, and so on for 50 weeks, the last payment being \$2.50. Show that the amount A of this increasing annuity at 4% exact simple interest is, where $w = 49$ weeks,

$$A = .05\,[(w+1) + w + (w-1) + (w-2) + \cdots + 2 + 1] +$$

$$\frac{(.05)(.04)(7)}{(365)(2)}\,[(w+1)w + w(w-1) + (w-1)(w-2) + \cdots + (3)(2) + (2)(1)]$$

$$= \frac{(.05)(w+2)(w+1)}{2} + \frac{(.05)(.04)7}{(365)(2)}\left[\frac{w(w+1)(w+2)}{3}\right]$$

$$= \$63.75 + .80$$

$$= \$64.55$$

CHAPTER III

APPLICATIONS

33. Introduction. In Chapter I interest and discount formulas, both simple and compound, have been derived for finding values of a single sum of money, and in Chapter II compound interest formulas have been derived for finding values of certain sets of sums, namely the annuities certain. Various applications of the formulas for a single sum have been given; also a few applications to problems involving sets of sums have been given. In this chapter the applications will be treated more fully and systematically, especially those that involve sets of sums. In these applications the equations in the unknowns to be found will usually be set up by means of the method discussed in Art. 16, Chapter I, that is, by equating values of sets of sums.

In solving many of the problems given in this chapter the following procedure will be found helpful: (1) specify clearly all sums, rates, and terms involved, both known and unknown; (2) note the sets having equal values; (3) set up the equations by equating values; (4) solve the equations and perform the computations. When the data are complex a line diagram may be used to advantage in visualizing the problem. In determining the sets having equal values it is advisable to view each problem as an actual business transaction. The solutions of the equations are based on the same processes that are used in the illustrative examples in the text of Chapters I and II.

The applications are arranged under three headings as follows: (1) Debts and Sinking Funds; (2) Investments; (3) Depreciation and Capitalized Costs.

DEBTS AND SINKING FUNDS

34. Methods of paying debts. Unless otherwise stated, the word "debt" will mean an interest-bearing debt. The interest payments on a debt are due at specified times, and the principal of the debt may be extinguished by a set of one or more payments. The extinction of a debt through a set of payments is called *amortization.* A schedule, like that in Example 1, Art. 16, Chapter I, which shows the payments on the principal, the interest payments, and the unpaid principal in the process of amortizing a debt is called an *amortization schedule.*

Different methods of amortizing a debt may be readily formulated by specifying different sets of payments on the principal. Those which follow occur frequently in practice and they suffice to illustrate the general processes. The unknown to be determined may be the set of payments of principal and interest, the term, or the rate. Methods for finding the first two of these types of unknowns are given in Arts. 35 to 43 inclusive; methods for finding an unknown rate are given in Arts. 56 to 60 inclusive.

35. The debt is retired in a known time by periodic payments, including interest and principal, which are equal in amount. In this method the equal payments, including interest and principal, constitute an annuity of unknown rent R having a given term n and $mr = 1$. The value of R can be found by solving the equation obtained by equating the present value of this annuity to the original principal P; this gives,

$$Ra_{\overline{mn}|\frac{j}{m}} = P,$$

or

$$R = \frac{P}{a_{\overline{mn}|\frac{j}{m}}}$$

EXAMPLE 1. Annuity bonds to the amount of $10,000, bearing interest at 5 per cent payable semi-annually, are to be repaid, interest and principal, in ten equal semi-annual instalments. Find the amount of each instalment and construct an amortization schedule.

SOLUTION. By the above formula,

$$R = \frac{10000}{a_{\overline{10}|.025}}$$
$$= \$1142.59$$

AMORTIZATION SCHEDULE

Year	Interest Period	Principal at Beginning of Period	Interest Due at End of Period	Payment of Interest and Principal	Principal Repaid
$\frac{1}{2}$	1	10,000.00	250.00	1142.59	892.59
1	2	9,107.41	227.69	1142.59	914.90
$1\frac{1}{2}$	3	8,192.51	204.81	1142.59	937.78
2	4	7,254.73	181.37	1142.59	961.22
$2\frac{1}{2}$	5	6,293.51	157.34	1142.59	985.25
3	6	5,308.26	132.71	1142.59	1009.88
$3\frac{1}{2}$	7	4,298.38	107.46	1142.59	1035.13
4	8	3,263.25	81.58	1142.59	1061.01
$4\frac{1}{2}$	9	2,202.24	55.05	1142.59	1087.54
5	10	1,114.70	27.87	1142.59	1114.72
Totals			1425.88	11425.90	10000.02

Exercise. Reconstruct the amortization schedule above by carrying all the work to mills, using $R = 1142.588$.

Example 2. A $10,000 debt is to be paid, interest and principal, in ten equal semi-annual instalments. Find the amount of each instalment if the unpaid principal bears 5 per cent payable semi-annually during the first three years, and 6 per cent semi-annually during the last two years.

Solution. Equating the value of the instalments at the end of three years to the value of the debt at the same date gives

$$R(s_{\overline{6}|.025} + a_{\overline{4}|.03}) = 10{,}000\ (1.025)^6$$

$$R = \frac{11596.934}{10.1048351}$$

$$= \$1147.66$$

Exercise. Construct an amortization schedule for Example 2. (Note that the interest rate per half year changes from .025 to .03 at the end of three years.)

EXERCISES

1. Annuity bonds to the amount of \$25,000 bearing interest at $5\frac{1}{2}\%$ payable annually are to be repaid, interest and principal, in nine equal annual instalments. Find the amount of each instalment and construct an amortization schedule. Ans. \$3595.99.

2. A cash payment of \$1500 is made on a house sold for \$6000. The balance is to be paid in 10 years by equal semi-annual payments which include interest and principal. If unpaid principal bears interest at 6% payable semi-annually, find the semi-annual payment and construct an amortization schedule.

3. A trust fund of \$125,000 is created to provide equal quarterly payments for 10 years, at the end of which time the fund is to be exhausted. Find the quarterly payment if unexpended portions of the fund bear interest at 5% payable quarterly. Ans. \$3990.18.

4. A balance of \$1200 on the purchase price of an automobile is repaid in equal instalments, including interest and principal, at the end of each month for one year. Find the monthly instalment if the debt bears interest at 12% payable monthly. Construct a schedule. Ans. \$106.62.

36. The debt is retired in a known time by periodic payments, including interest and principal, which are nearly equal in amount. In some transactions in which it would be desirable to make equal periodic payments of interest and principal the conditions are such that it is impossible to do so. For example, in paying a debt in a given time represented by bonds of given denomination it is not possible to make equal payments of interest and principal since the interest must be paid in full and the payments on the principal must be integral multiples of the redemption value of the bonds. In such cases the method in Art. 35 can be used to find the amount of the periodic payment under the assumption that this is uniform, and then the difference between this amount and the interest due at the end of any period determines the number of bonds that should be retired to make the payments of interest and principal nearly equal.

Example. A debt of \$10,000 consisting of 100 bonds of \$100 denomination, bearing interest at 5% payable semi-annually, is to be paid, interest and principal, in ten semi-annual instalments as nearly equal as possible. Construct an amortization schedule.

Solution. If the instalments were equal in value, the amount of each instalment would be \$1142.59 by Example 1, Art. 35. The interest due in a half year is \$250; the difference, $1142.59 - 250 = 892.59$, shows that 9 bonds

should be retired at this time. The new principal is \$10,000 − \$900 = \$9100 and the interest due at the end of one year is \$227.50; the difference, 1142.59 − 227.50 = 915.09, shows that 9 bonds should be retired at the end of one year. Continuing this process gives the following:

AMORTIZATION SCHEDULE

Year	Interest Period	Principal at Beginning of Period	Interest Due at End of Period	Number of Bonds Retired at End of Period	Principal Repaid at End of Period	Total Payment of Interest and Principal
$\frac{1}{2}$	1	10000	250.00	9	900	1150.00
1	2	9100	227.50	9	900	1127.50
$1\frac{1}{2}$	3	8200	205.00	9	900	1105.00
2	4	7300	182.50	10	1000	1182.50
$2\frac{1}{2}$	5	6300	157.50	10	1000	1157.50
3	6	5300	132.50	10	1000	1132.50
$3\frac{1}{2}$	7	4300	107.50	10	1000	1107.50
4	8	3300	82.50	11	1100	1182.50
$4\frac{1}{2}$	9	2200	55.00	11	1100	1155.00
5	10	1100	27.50	11	1100	1127.50
Totals			1427.50	100	10000	11427.50

EXERCISES

1. A city issues street improvement bonds to the amount of \$25,000 in denominations of \$500, bearing interest at $5\frac{1}{2}\%$ payable semi-annually. Construct a schedule showing the number of bonds to be retired each half year for five years in order to make the payments of interest and principal as nearly equal as possible.

2. A \$10,000 issue of bonds in denominations of \$1000, bearing interest at 5% payable semi-annually, is to be retired in eight semi-annual instalments as nearly equal as possible. Construct an amortization schedule.

3. A \$100,000 issue of bonds, bearing interest at 6% payable annually and consisting of 250 bonds of \$100, 50 bonds of \$500, and 50 bonds of \$1000, denomination, is to be retired, interest and principal, in ten annual instalments as nearly equal as possible. Construct an amortization schedule.

37. The debt is retired by payments, including interest and principal, all of which are equal and known except the last, which is equal to or less than the known payment. In Arts. 35 and 36 methods of payment of debts are presented in which the number of periodic payments of interest and principal is given and the problem is to determine the payments. In the method presented in this article a given amount is paid periodically until the debt is liquidated, or, as is usually the case, until the unpaid principal becomes less than the given payment; the problem is to find the number of the given payments and the amount of the last payment. The solution of Example 4, Art. 25, Chapter II shows how these quantities can be found. Another illustration is given by the

EXAMPLE. If instalments of \$100, including interest and principal, are paid at the end of each month on a debt of \$10,000, bearing interest at 5% payable monthly, find how many instalments are needed before the unpaid principal is less than \$100; also find the amount needed to pay off the balance of the debt at the end of the following month.

SOLUTION. By formula (2), Art. 3, Chapter II,

$$10000 = 100 \frac{1 - \dfrac{1}{\left(1 + \frac{.05}{12}\right)^{12n}}}{\dfrac{.05}{12}}$$

Solving, $$\left(1 + \frac{.05}{12}\right)^{12n} = \frac{12}{7}$$

$$12\,n = \frac{\log 12 - \log 7}{\log 12.05 - \log 12} = 129.6$$

This result indicates that 129 monthly instalments of \$100 each are needed to reduce the debt to an amount less than \$100. Let x be the additional amount needed at the end of 130 months to extinguish the debt. Equating the value of the payments to the value of the debt at the end of 130 months gives

$$x + 100\, s_{\overline{130}|\frac{.05}{12}} - 100 = 10000\left(1 + \frac{.05}{12}\right)^{130}$$

$$x = \$62.90 \quad \text{Tables III and V.}$$

EXERCISE 1. Find x by equating the present value of the payments to \$10,000.

EXERCISE 2. What amount is needed to extinguish the debt at the end of 129 months? [Discount for one month the value of x found in the above

solution; also solve by equating the value of the payments to that of the debt at the end of 129 months.]

EXERCISE 3. Find n by interpolation from Table VI.

This method of paying debts is frequently used in repaying loans made by building and loan associations. The number of payments and the value of the last payment can be found by constructing an amortization schedule, and this is done by the association in keeping its accounts. The advantage of the solution given above lies in the fact that it requires much less computation than is needed to make an amortization schedule.

EXERCISES

1. On a \$6000 mortgage, bearing interest at 6% payable semi-annually, payments of \$360 were made at the end of each six months. Find the number of payments which will reduce the unpaid principal to less than \$360; also find the amount needed to pay off the remainder of the debt at the end of the next six months. Ans. 23; \$163.24.

2. Same as Exercise 1, except that the interest rate is 7% payable semi-annually.

3. A trust fund of \$100,000 is invested at ($j = .05$, $m = 2$). From the fund semi-annual payments of \$3500 are to be made as long as possible. Find the number of full payments and the amount of the last one.

Ans. 50; \$2578.54.

4. On a \$3000 mortgage payments of \$200 are made at the end of each six months. If the mortgage bears interest at 7% payable semi-annually during the first three years, and at 6% payable semi-annually after three years, find the number of payments and the amount of the last payment.

Ans. 20; \$184.65.

38. The debt is retired by equal known periodic payments on the principal. In this method the number of payments is found by dividing the principal by the amount of each payment, and the interest payment for each period can be readily computed since the principal on which interest is computed decreases by a known amount each time a payment on the principal is made. This method includes the case in which the principal is extinguished by a single payment as in an ordinary bond.

EXAMPLE. A debt of \$10,000, bearing interest at 5% payable semi-annually, is retired in 5 years by semi-annual payments of \$1000 on the principal. Construct an amortization schedule.

AMORTIZATION SCHEDULE

Year	Interest Period	Principal at Beginning of Period	Interest Due at End of Period	Payment on Principal	Total Payment
$\frac{1}{2}$	1	10000	250	1000	1250
1	2	9000	225	1000	1225
$1\frac{1}{2}$	3	8000	200	1000	1200
2	4	7000	175	1000	1175
$2\frac{1}{2}$	5	6000	150	1000	1150
3	6	5000	125	1000	1125
$3\frac{1}{2}$	7	4000	100	1000	1100
4	8	3000	75	1000	1075
$4\frac{1}{2}$	9	2000	50	1000	1050
5	10	1000	25	1000	1025

EXERCISES

1. A city issues $20,000 in bonds of $1000 denomination, bearing interest at $5\frac{1}{2}\%$ payable semi-annually. If two bonds are retired at the end of each half year, construct a table showing the amortization of the debt.

2. A farm is purchased for $10,000, one half of which is paid in cash. The purchaser agrees to pay $1000 at the end of each year on the unpaid principal of the debt. Construct an amortization schedule if the debt bears interest at 7% payable annually.

3. Same as Exercise I, except that the interest rate is 6% payable semi-annually.

39. The debt is retired by payments which may be irregular both in amount and in time. An illustration of this method has been given in Example 1, Art. 16, Chapter I. Another illustration is afforded by the

Example. A note for $500, issued January 1, 1915, stipulated that unpaid principal was to bear interest at 6% payable semi-annually and that unpaid interest was to bear interest at 8% payable semi-annually. Interest payments were made during the first year and a payment of $100, including interest and principal, was made January 1, 1918. If no further payments were made, find the amount due January 1, 1923.

SOLUTION. The interest due semi-annually during the first three years was \$15 and the unpaid principal January 1, 1918, just after the \$100 payment, was

$$500 + 15\, s_{\overline{4}|.04} - 100 = 400 + 15\, s_{\overline{4}|.04}$$

The interest due semi-annually during the last five years was $.03(400+15\, s_{\overline{4}|.04})$ and the total amount due January 1, 1923 was

$$\begin{aligned} 400 + 15\, s_{\overline{4}|.04} + .03(400 + 15\, s_{\overline{4}|.04})s_{\overline{10}|.04} &= (400 + 15\, s_{\overline{4}|.04})(1 + .03\, s_{\overline{10}|.04}) \\ &= \$630.71 \end{aligned}$$

EXERCISES

1. Solve the above example if the payment made January 1, 1918, was \$50.

2. A note for \$500 issued January 1, 1916 stipulated that unpaid principal was to bear 6% interest payable semi-annually and that unpaid interest was to bear 7% payable semi-annually. Payments on principal and interest were made as follows: January 1, 1917, \$100; January 1, 1918, \$50; and July 1, 1920, \$200. Find amount due January 1, 1924. Ans. \$334.84.

3. On a debt of \$5000 payments of \$200 were made at the end of each half year for 4 years, after which semi-annual payments of \$300 were made until the debt was reduced to less than \$300. For the first two years the interest rate was 7% payable semi-annually; for the remainder of the term it was 6% payable semi-annually. Find the amount of the last payment if it was made six months after the debt was reduced to less than \$300. Ans. \$52.69.

40. The unpaid principal at any time. The unpaid principal at any specified time of a debt paid by instalments which are known can evidently be determined by constructing an amortization schedule to this time. For example, the schedule for Example 1, Art. 35, shows that the unpaid principal just after the sixth instalment is paid is \$4298.38. Two other methods can often be used to find the unpaid principal at any time; one of these may be called the *prospective* method, and the other the *retrospective*. In the prospective method the unpaid principal just after an instalment is paid is viewed as the value at that time of the subsequent instalments of interest and principal. In the retrospective method the unpaid principal just after any instalment is paid is viewed as the difference between the value of the original principal at the time and that of the prior instalments of interest and principal. If P_k denotes the unpaid principal just after the kth instalment, an application of these methods to the method of paying debts

treated in Art. 35 gives respectively:

$$P_k = Ra_{\overline{mn-k}|\frac{j}{m}} \qquad \text{(prospective)}$$

$$P_k = P\left(1+\frac{j}{m}\right)^k - Rs_{\overline{k}|\frac{j}{m}} \qquad \text{(retrospective)}$$

EXERCISE 1. Compute the right-hand member of each of these expressions when $k = 6$ for Example 1, Art. 35.

EXERCISE 2. Compute the unpaid principal in Example 3, Art. 35, at the end of the fourth year, by both the prospective and the retrospective methods.

EXERCISES

1. Use the prospective and the retrospective methods to find the unpaid principal at the beginning of the third year for the annuity bonds in Example 1, solved in Art. 35. Ans. \$5308.26.

2. Same as Exercise 1, for the example solved in Art. 37. Ans. \$7739.39.

3. A house is bought for \$7800, of which \$2000 is paid in cash. On the balance which bears interest at 6%, payable semi-annually, payments of \$390 were made at the end of each six months. Immediately after the semi-annual payment which reduced the debt to less than \$4000 it was refunded. Find the amount of the debt when it was refunded. Ans. \$3879.26.

41. Sinking funds. A sinking fund consists of a set of sums put aside for the purpose of meeting a future obligation. The sums put aside should be productively invested. The future obligation is often an instalment on a debt, a sum of money needed to restore capital invested, or a sum of money needed to replace an article which is subject to depreciation. In the latter case the sinking fund is sometimes called a depreciation fund. When the sums are put aside at periodic intervals, they constitute an annuity and the annuity formulas of Chapter II may be used with such sinking funds. Unless otherwise stated sinking funds having equal periodic payments will be understood in what follows.

In sinking-fund operations the unknowns are ordinarily the sums of money put aside periodically; the interest rate and the term are usually known. When the sums are equal, the value of each can be found by solving formula (1) or (3), Chapter II, for R. For example, when $mr = 1$, formula (3) gives

$$R = V_n \cdot \frac{1}{s_{\overline{mn}|\frac{j}{m}}}$$

where V_n is the amount to be accumulated by the sinking fund, n is the term of years, and j is the nominal interest rate converted m times per year.

The amount V_{n_1} in the sinking fund at the end of n_1 years $(n_1 < n)$ can also be found by formula (3).

$$V_{n_1} = Rs_{\overline{mn_1}|\frac{j}{m}} = V_n \cdot \frac{1}{s_{\overline{mn}|\frac{j}{m}}} \cdot s_{\overline{mn_1}|\frac{j}{m}}$$

Example. A sinking fund was created to pay the principal of a loan of \$100,000 in 5 years by equal semi-annual instalments. If the sinking fund accumulates at $(j = .05,\ m = 2)$, find (1) the semi-annual instalment and (2) the amount in the sinking fund at the end of three years.

Solution. By the above formulas

$$R = 100{,}000 \cdot \frac{1}{s_{\overline{10}|.025}} = \$8925.876$$

$$V_3 = \$8925.876 \cdot s_{\overline{06}|.025} = \$57{,}016.146$$

Exercise. Check the values found for R and V_3 by completing the following

SINKING–FUND SCHEDULE

Year	Interest Period	Payment to Fund at End of Period	Interest Due on Fund at End of Period	Amount in Fund at End of Period
½	1	8925.876		8925.876
1	2	8925.876	223.147	18174.899
1½	3	8925.876	454.372	27555.147
2	4	8925.876		
2½	5	8925.876		
3	6	8925.876		
3½	7	8925.876		
4	8	8925.876		
4½	9	8925.876		
5	10	8925.876		

EXERCISE

A debtor creates a sinking fund to pay the principal of a debt of \$6178.75 by depositing \$120 at the end of each three months in a savings bank which pays 5% converted quarterly. Show that the deposits must be made for 10 years. Construct a sinking fund schedule.

Sinking funds to restore invested capital and to replace articles subject to depreciation are considered in Arts. 47, 50, and 64 respectively. Sinking funds to repay loans are considered in Art. 42.

42. The sinking-fund method of retiring a debt. In this method a sinking fund is created to retire the principal of the debt when due; the interest payments are made separately. When the payments into a sinking fund created to meet a debt are made at the times of the interest payments on the debt and when the sinking-fund payments earn the same interest rate as that borne by the debt, the sinking-fund payment plus the interest payment on the debt is the same as the periodic payment of interest and principal necessary to pay the debt (Art. 35). If P denotes the principal of the debt, the sinking-fund payment, found by replacing V_n by P in the above formula, is $\frac{P}{s_{\overline{mn}|\frac{j}{m}}}$, and the interest payment is $P \cdot \frac{j}{m}$. The sum of these two payments is $P\left(\frac{1}{s_{\overline{mn}|\frac{j}{m}}} + \frac{j}{m}\right) = P \cdot \frac{1}{a_{\overline{mn}|\frac{j}{m}}}$ (Formula 5, Chapter II), which by the method in Art. 35 is the periodic payment of interest and principal. When, however, the sinking-fund payments bear the rate j' and the debt bears the rate j each payable m times per year, the sinking-fund payment plus the interest payment is $P\left(\frac{1}{s_{\overline{mn}|\frac{j'}{m}}} + \frac{j}{m}\right)$

EXAMPLE 1. Find the semi-annual payment into a sinking fund which accumulates at 5% converted semi-annually, necessary to pay a debt of \$10,000 due in 5 years and bearing interest at 5% payable semi-annually.

SOLUTION. By Art. 41, the sinking-fund payment, R, is given by

$$R = 10{,}000 \frac{1}{s_{\overline{10}|.025}} = \$892.588$$

It may be noted that this sinking-fund payment plus the interest payment of \$250 is \$1142.588, as found in Example 1, Art. 35.

EXERCISE 1. Find the amount in the sinking fund just after the sixth payment is made.

EXERCISE 2. Find the actual indebtedness just after the sixth sinking-fund payment is made by deducting the amount found in Exercise 1 from the original principal, $10,000.

EXAMPLE 2. Solve Example 1 if the sinking-fund payments are accumulated at 4% converted semi-annually.

SOLUTION. By Art. 41 the sinking-fund payment is

$$R = 10{,}000 \cdot \frac{1}{s_{\overline{10}|.02}} = \$913.265$$

In this case the sinking-fund payment plus the interest payment is $1163.265.

EXERCISE 1. Find the amount in the sinking fund just after the sixth payment is made.

EXERCISE 2. Find the actual indebtedness just after the sixth sinking-fund payment is made by deducting the amount found in Exercise 1 from the original principal $10,000.

EXAMPLE 3. The principal of a $10,000 debt bearing interest at 5% payable semi-annually is to be paid in two $5000 instalments, the first in 5 years and the second in 8 years. Deposits are made semi-annually into a sinking fund which pays 4% converted semi-annually, for the purpose of meeting these instalments when due. If the deposits are such that the sum of the deposit and the interest due at any time is constant, find the amount of each deposit.

SOLUTION. If R denotes the semi-annual deposit during the first 5 years, then $R + 125$ is the semi-annual deposit during the last 3 years, since interest payments decrease by $125 at the end of 5 years. Equating the amount of the deposits to the amount of the instalments gives

$$\begin{aligned} Rs_{\overline{16}|.02} + 125\, s_{\overline{6}|.02} &= 5000 + 5000\,(1.02)^6 \\ R &= \$528.04 \\ R + 125 &= \$653.04 \end{aligned}$$

EXERCISES

1. Find the annual payment into a sinking fund which accumulates at ($j = .055$, $m = 1$) necessary to pay a debt of $25,000 due in 9 years, and bearing interest at $5\frac{1}{2}$% payable annually. Find also the sum of the annual interest on the debt and the sinking-fund payment. (See Ex. 1, Art. 35.)

Ans. $2220.99; $3595.99.

2. Same as Exercise 1 except that the sinking fund accumulates at ($j = .045$, $m = 1$).

3. Find the quarterly payment into a sinking fund which accumulates at ($j = .04$, $m = 4$) necessary to pay a debt of $100,000 due in 10 years, and bearing interest at 6% payable quarterly. Find also the sum of the interest and sinking-fund payments. Ans. $2045.56; $3545.56.

4. Find the semi-annual payment into a sinking fund which accumulates at ($j = .045$, $m = 2$) necessary to pay the principal of the following debts: (1) \$1000 due in 1 year and bearing interest at 4% payable semi-annually; (2) \$2000 due in 2 years and bearing interest at 5% payable semi-annually; and (3) \$3000 due in three years and bearing interest at 6% payable semi-annually. Find also the sums of the interest and sinking-fund payments at the end of each half year.

Ans. \$974.21; \$1134.21; \$1134.21; \$1114.21; \$1114.21; \$1064.21; \$1064.21.

5. Same as Exercise 4, except that the sinking-fund payments are such that the sum of the sinking-fund payments and the interest due at the end of each six months is constant. Construct a sinking-fund schedule.

Ans. \$945.25; \$945.25; \$965.25; \$965.25; \$1015.25; \$1015.25; \$1105.25.

43. Book Values. In keeping accounts connected with the payments of debts the unpaid principal of a debt at any entry date is often called the *Book Value* of the debt at that date. When a sinking fund is created to meet the debt when due, the book value of the debt at each entry date is its original value less the amount in the sinking fund. "Book values" are also used in connection with other items whose values change from entry to entry. For example, in Art. 46 the book values of an investment are found at different entry dates, and in Art. 62 the book values of an article subject to depreciation are determined. The book values of a debt whose principal is paid by means of a sinking fund are illustrated by the

Example. The principal of a \$10,000 debt bearing interest at 5% payable semi-annually is to be paid at the end of 5 years by means of equal semi-annual deposits into a sinking fund which pays 4% converted semi-annually. Construct a schedule showing the book value of the indebtedness at the end of each half year.

Solution. By Example 2, Art. 42, the sinking fund payment is \$913.265.

SCHEDULE

Year	Interest Period	Book Value of Debt at End of Period	Payment to Sinking Fund at End of Period	Interest Due on Fund at End of Period	Amount in Fund at End of Period
0	0	10000			
$\frac{1}{2}$	1	9086.735	913.265		913.265
1	2	8155.205	913.265	18.265	1844.795
$1\frac{1}{2}$	3	7205.044	913.265	36.896	2794.956
2	4	6235.880	913.265	55.899	3764.120
$2\frac{1}{2}$	5	5247.333	913.265	75.282	4752.667
3	6	4329.015	913.265	95.053	5760.985
$3\frac{1}{2}$	7	3210.530	913.265	115.220	6789.470
4	8	2161.476	913.265	135.789	7838.524
$4\frac{1}{2}$	9	1091.441	913.265	156.770	8908.559
5	10	.005	913.265	178.171	9999.995

EXERCISES

1. Construct a schedule showing the book value of the debt at the end of each year for Exercise 1, Art. 42.

2. Construct a schedule showing the book value of the debt at the end of each year for Exercise 2, Art. 42.

3. Construct a schedule showing the book value of the total indebtedness for Exercise 5, Art. 42.

INVESTMENTS (BONDS, STOCKS, NOTES, SAVINGS)

44. The problems of investment. A problem in investment, like one in debts, may have a sum of money, a term of years, or a rate for an unknown. When a sum of money is unknown, it is usually the purchase price or value of an investment, the investment rate and the term being known. The value of an investment under a given rate depends upon the set of sums which the investment returns to the investor. The method for finding the value

of an investment depends, then, upon the nature of the return. In Arts. 45 to 54 inclusive various types of return are considered. When a term of years is unknown, it is frequently the time required to pay for an investment bought on the instalment plan. Building and Loan Stocks afford a good illustration. This type of problem is treated in Art. 55. The rate, or yield, is frequently unknown in problems in investments, debts, and other fields of finance. Methods for finding rates are given in Arts. 56 to 60 inclusive.

A general principle in investments is that the capital invested should not be impaired. That is, the amount invested should be returned to the investor for reinvestment during, or at the end of, the term of the investment.

45. The value of an investment whose return is a sum of money. This simple problem has been treated quite fully in Chapter I. It is the problem of finding the present value of a sum of money due at some future time. The discounting of notes by banks affords a good illustration. Any one of the four formulas for finding the present value of a sum may be used to solve problems of this type.

46. The value of an investment whose return is an annuity of fixed rent. Annuity bonds furnish good illustrations of this type of return. They are bonds which are repaid in instalments such that the interest due plus the amount paid on the principal or *face value* is constant or as nearly constant as the denomination of the bonds permit. In this article the case is treated in which the instalments are equal; in article 54, the case is treated in which the instalments are nearly equal. When the purchase price of a bond exceeds its face value by an amount P, the bond is said to be bought at a *premium* P. When the face value of the bond exceeds its purchase price by an amount D, the bond is said to be bought at a *discount* D.

The return in this type of investment being an annuity, the purchase price is the present value of the annuity at the interest rate desired by the investor. For example, if an annuity of rent R, term n years, and rent period $\frac{1}{m}$ years, is purchased to yield the

investor j per cent payable m times per year, the purchase price V is given by

$$V = Ra_{\overline{mn}|\frac{j}{m}}$$

EXAMPLE. Annuity bonds to the amount of \$10,000 bearing 5% payable semi-annually are to be repaid, interest and principal, in 10 equal semi-annual instalments. Find the purchase price to yield the investor 4% converted semi-annually.

SOLUTION. By the example in Art. 35, the debtor makes semi-annual payments of \$1142.588 for a term of 5 years. These payments constitute the return to the investor; their present value at 4% semi-annually gives for the purchase price, $V = 1142.588\, a_{\overline{10}|.02} = \$10{,}263.394$. In this case the bond is purchased at a premium of \$263.39.

EXERCISE 1. Solve this example if the bonds are bought to yield 6% payable semi-annually. What is the discount?

EXERCISE 2. Solve this example if the bonds are bought to yield 4% payable annually. What is the premium?

EXERCISE 3. What is the purchase price of the bonds if bought 3 years before they mature, to yield 4% payable semi-annually. What is the purchase price if bought 2 years before they mature?

If the return is composed of two or more annuities, each of fixed rent, the value of the investment is the sum of the values of the component annuities.

In investments of this type each payment received from the return includes both interest on the investment and repayment of a portion of the capital invested. The payments on the capital may be put into a sinking fund, to restore the capital invested at the end of the term; this plan is discussed in Art. 47. Usually, however, in keeping accounts these payments are deducted in turn from the capital invested so that the book value of the investment changes each time a payment is received, to correspond to the purchase price at the time of entry. This change in purchase price is illustrated by the above example and by Exercise 3. In this method of keeping accounts the payments on the capital invested are available for reinvestment as soon as they are received. For the above example the following is an

INVESTMENT SCHEDULE

Year	Interest Period	Book Value at Beginning of Period	Interest Due on Book Value at End of Period	Payment Received at End of Period	Amount Repaid on Capital Invested
$\frac{1}{2}$	1	10263.394	205.268	1142.588	937.320
1	2	9326.074	186.521	"	956.067
$1\frac{1}{2}$	3	8370.007	167.400	"	975.188
2	4	7394.819	147.896	"	994.692
$2\frac{1}{2}$	5	6400.127	128.003	"	1014.585
3	6	5385.542	107.711	"	1034.877
$3\frac{1}{2}$	7	4350.665	87.013	"	1055.575
4	8	3295.090	65.902	"	1076.686
$4\frac{1}{2}$	9	2218.404	44.368	"	1098.220
5	10	1120.184	22.404	"	1120.184
Totals			1162.486	11425.880	10263.394

Exercise 4. Construct an investment schedule for Exercise 1.

When the term of the annuity which constitutes the return becomes infinite, that is, when the return is a perpetuity, the purchase price is the present value of a perpetuity. When for example the rent is R payable m times per year, the purchase price of the perpetuity, to yield j payable m times per year is (Art. 29, Chapter II)

$$V = \frac{R}{\frac{j}{m}}$$

EXERCISES

1. Annuity bonds to the amount of $100,000 bearing interest at $5\frac{1}{2}\%$ payable quarterly are to repaid, interest and principal, in 40 equal quarterly instalments. Find the purchase price to yield the investor 6% converted quarterly. Construct an investment schedule. Ans. $97733.00.

2. In Exercise 1, find the purchase price to yield the investor 6% converted semi-annually.

3. The return on a lease is \$1000 at the end of each three months. The lease was sold 5 years before the end of its term. Find the purchase price to yield the investor 5% converted semi-annually. Ans. \$17612.10.

4. The return on a 99 year lease is \$5500 at the end of each six months. It is purchased 5 years after it was executed. Find the purchase price to yield the investor 6% converted semi-annually. Ans. \$182806.69.

5. Same as Exercise 4, except that the return on the lease is \$5500 semi-annually for the first 10 years, \$6500 semi-annually for the next 10 years, and \$7000 semi-annually for the remainder (79 years) of its term.

6. Solve Exercise 4 if the semi-annual return of \$5500 is a perpetuity.
Ans. \$183333.33.

7. The return on a lease on a storeroom is \$150 at the end of each month for the first 5 years, and \$200 a month for the next 5 years. The lease was sold one year after it was executed to yield its owner 8% converted annually. Find the sale price. Ans. \$13475.49.

47. The value of an investment whose return is an annuity when a sinking fund is created to restore the capital invested. If the payments on the capital invested are used to create a sinking fund to restore the capital at the end of the investment term instead of being used to reduce the book value of the investment, the amount of the investment may be found by equating the sinking-fund payment plus the interest on the investment to the rent of the annuity constituting the return. When the sinking-fund rate equals the investment rate the interest on the investment for the case treated in Art. 46 is $V \cdot \frac{j}{m}$ payable m times per year, and the sinking-fund payment needed to restore the purchase price V in n years is $V \cdot \frac{1}{s_{\overline{mn}|\frac{j}{m}}}$ (Art. 41). Hence in this case

$$V \cdot \frac{1}{s_{\overline{mn}|\frac{j}{m}}} + V \cdot \frac{j}{m} = R$$

or

$$V \frac{1}{a_{\overline{mn}|\frac{j}{m}}} = R \qquad \left(\text{since } \frac{1}{a_{\overline{mn}|\frac{j}{m}}} = \frac{1}{s_{\overline{mn}|\frac{j}{m}}} + \frac{j}{m}\right)$$

$$V = Ra_{\overline{mn}|\frac{j}{m}}$$

When the sinking-fund rate is j' converted m times per year, the purchase price is determined by the equation

$$V \cdot \frac{1}{s_{\overline{mn}|\frac{j'}{m}}} + V \cdot \frac{j}{m} = R$$

$$V = \frac{R}{\dfrac{1}{s_{\overline{mn}|\frac{j'}{m}}} + \dfrac{j}{m}}$$

Example. It is estimated that a mine will yield a net return of \$10000 at the end of each year for 20 years, when it will be exhausted. Find the purchase price to yield 10% converted annually on the investment if a sinking fund which accumulates at 4% converted annually is created to restore the capital invested in 20 years?

Solution. In this example the annual interest payment is $V(.1)$ and the annual sinking-fund payment is $V \cdot \frac{1}{s_{\overline{20}|.04}}$, so that

$$V \frac{1}{s_{\overline{20}|.04}} + V(.1) = 10000$$

Solving, $$V = \frac{10000}{\dfrac{1}{s_{\overline{20}|.04}} + .1} = \frac{10000}{.1335818} = \$74860.50$$

EXERCISES

1. It is estimated that a mine will yield a net return of \$1000 at the end of each year for 35 years, when it will become exhausted. Find the purchase price to yield 8% payable annually on the investment if a sinking fund which accumulates at 5% converted semi-annually is created to restore the capital invested in 35 years. Ans. \$10997.57.

2. It is estimated that a taxicab will yield a net return of \$500 at the end of each year for 4 years, after which it will be worthless. Find the purchase price to yield 10% payable annually on the investment if a sinking fund which accumulates at 4% converted annually is created to restore the capital invested in 4 years. Ans. \$1490.36.

3. It is estimated that the return from an oil well will average \$10000 annually for 3 years, after which it will be worthless. Find its value at the time of its development if the investment is to pay 8% payable annually and a sinking fund which accumulates at 5% converted annually is created to restore the capital invested in three years.

48. The value at date of issue or at an interest payment date of an ordinary bond. The return is a single sum and an annuity. Ordinary bonds furnish the best illustration of this type of return. In an ordinary bond a promise is made to pay a specified amount, called the *redemption value*, at the end of a stated term, and to pay interest at periodic intervals during this term on a specified value called the *face value* of the bond. The redemption value is usually the same as the face value; sometimes, however, to make the bond more attractive to buyers the redemption value is greater than the face value. In the former case the bond is said to be redeemed at *par;* in the latter, it is said to be redeemed *above par*. The words "premium" and "discount" for an ordinary bond are defined as in the preceding article for an annuity bond; that is, the premium is the purchase price less the face value and the discount is the face value less the purchase price.

The purchase price or value of an investment, whose return is an annuity and a single sum, is the present value of the annuity plus the present value of the single sum under the interest rate desired by the investor. For example, in the case of an ordinary bond whose

face value is C
redemption value is F
bond rate is g payable m times per year
investment rate is j payable m times per year
term is n years

the value, V, is $V = F\left(1+\frac{j}{m}\right)^{-mn} + \frac{Cg}{m} a_{\overline{mn}|\frac{j}{m}}$

This formula can be written also in the two forms

$$V = F + \left(\frac{Cg}{m} - \frac{Fj}{m}\right) a_{\overline{mn}|\frac{j}{m}}$$

$$V = \frac{Cg}{j} + \left(F - \frac{Cg}{j}\right)\left(1+\frac{j}{m}\right)^{-mn}$$

by use of the relation $a_{\overline{mn}|\frac{j}{m}} = \dfrac{1-\left(1+\frac{j}{m}\right)^{-mn}}{\frac{j}{m}}$ (Formula 2_1, Chapter II)

To get the second formula from the first, replace $\left(1+\frac{j}{m}\right)^{-mn}$ by $1-\frac{j}{m}\,a_{\overline{mn}|\frac{j}{m}}$; to get the third from the first, replace $a_{\overline{mn}|\frac{j}{m}}$ by $\dfrac{1-\left(1+\frac{j}{m}\right)^{-mn}}{\frac{j}{m}}$. It may be noted that both interest and annuity tables are needed for the first formula, annuity tables only for the second, and interest tables only for the third. Excellent bond tables which give the values of V for various values of $\frac{j}{m}$ and n have been constructed.

When $F = C$, the second form shows that the bond is bought at a premium when $g > j$, and at a discount when $g < j$. If P and D denote the premium and discount respectively, then

$$P = C\frac{g-j}{m}a_{\overline{mn}|\frac{j}{m}},\quad D = C\frac{j-g}{m}a_{\overline{mn}|\frac{j}{m}}$$

The student is advised against substituting into any one of these formulas in finding the value of a bond; it is better to make direct use of the principles needed to write them. These principles are seen at once by expressing the formulas verbally. For example, by the first formula, the value of a bond at a given investment rate equals the present value of the redemption value of the bond plus the present value of the annuity composed of the interest payments on the bond. These three principles will now be applied in turn to solve the

Example. A \$10000 bond issued July 1, 1923, bearing interest at 5% payable semi-annually, is to be redeemed at par in 5 years. Find its purchase price to yield 4 per cent converted semi-annually. The return on this investment consists of \$250 interest every half year for 5 years and \$10000 at the end of 5 years.

Solution 1. The purchase price of the bond is the present value at 4% converted semi-annually of \$10000 due in 5 years plus the present value at 4% converted semi-annually of the interest payments on the bond. That is,

$$\begin{aligned} V &= 10000(1.02)^{-10} + 250\,a_{\overline{10}|.02} \\ &= 8203.483 + 2245.646 = \$10449.129 \end{aligned}$$

SOLUTION 2. A \$10000 investment for 5 years at 4% payable semi-annually would yield for its return \$200 every half year for 5 years and \$10000 at the end of 5 years. The return on the bond exceeds this return by an annuity whose rent is \$50, whose term is 5 years and whose rent period is ½ year. Hence to find the purchase price of the bond the \$10000 investment must be increased by the present value of this annuity whose rent is \$50. That is,

$$V = 10000 + 50\, a_{\overline{10}|\,02}$$
$$= 10000 + 449.129 = \$10449.129$$

SOLUTION 3. A \$12500 investment for 5 years at 4% payable semi-annually would yield for its return \$250 every half year and \$12500 at the end of 5 years. This return exceeds the return on the bond by \$2500 due in 5 years. Hence to find the purchase price of the bond the \$12500 investment must be decreased by the present value of \$2500 due in 5 years. That is,

$$V = 12500 - 2500(1.02)^{-10}$$
$$= 12500 - 2050.871 = \$10449.129$$

EXERCISES

1. Find the value of the bond in the example above by each of the three methods, if the investment rate is $(j = .06, m = 2)$. Ans. \$9573.49.

2. Find the value of the bond in the example above 4 years, 3 years, 2 years, and 1 year before it matures.

3. Find the value of the bond in the example above if the redemption value is \$11000, the face value remaining at \$10000. Ans. \$11269.48.

4. Find the purchase price of a \$5000 bond bearing interest at 5½% payable semi-annually, if it is redeemable at par in 10 years, and if the investment yields $(j = .05, m = 2)$; if it yields $(j = .06, m = 2)$. Find the premium in the first instance and the discount in the second. Ans. \$194.86; \$185.97.

5. Same as Exercise 1, except that the term of the bond is 20 years.
Ans. \$8844.261.

6. A \$20000 bond bears interest at 5½% payable semi-annually and matures at 102 in 5 years. Find the purchase price to yield $(j = .05, m = 2)$; to yield $(j = .06, m = 2)$. Ans. \$20750.08; \$19871.13.

7. A \$100000 issue of bonds bearing interest at 5% payable semi-annually, and whose term is 10 years, was sold for \$106554.19 to yield the purchaser $(j = .045, m = 2)$. Find the redemption value of the issue.

8. If a sinking fund at the rate $(j = .04, m = 2)$ is used to restore the premium for the example solved in the text, find the semi-annual deposit.
Ans. \$41.02.

9. It is estimated that a taxicab will yield a net return of \$500 at the end of each year for 4 years, at which time it will have a scrap value of \$100. Find the purchase price to yield 8% annually.

10. It is estimated that the income of a lumber company in developing a tract of wooded land will be $10000 annually for 15 years, at which time production will cease and the land will be worth $2500. Find the purchase price correct to dollars to yield 10% annually. Ans. $76659.

49. Investment schedules for ordinary bonds. The usual method for keeping accounts for an ordinary bond investment is like that used for an annuity bond; each book value entry is the actual value of the bond, *at the investment rate*, at the time the entry is made. These values may be found by application of the principles used in solving the example and exercises in Art. 48. When a complete schedule is to be constructed, each new book value can be found more easily, however, by changing the value last entered by an amount which is the difference between the interest payment on the bond and the interest payment on the book value. This difference is subtracted each time when the purchase price is greater than the redemption value of the bond, and it is added when the purchase price is less than the redemption price. In the former instance the book value is said to be *written off* to the redemption value; in the latter it is said to be *written on*. The total amount to be written off, to make the book value agree with the redemption value, is, of course, the purchase price less the redemption value; the total amount to be written on, is the redemption value less the purchase price. In a schedule, the separate amounts written off are entered under the heading "For Amortization"; the amounts written on are entered under the heading "For Accumulation." When $F = C$, the amount amortized is the premium, and the amount accumulated is the discount.

Date	Book Value	Interest on Book Value at 2%	Dividend Received	For Amortization of Premium
July 1, 1923	10449.13			
Jan. 1, 1924	10408.11	208.98	250.00	41.02
July 1, 1924	10366.27	208.16	250.00	41.84
Jan. 1, 1925	10323.60	207.33	250.00	42.67
July 1, 1925	10280.07	206.47	250.00	43.53
Jan. 1, 1926	10235.67	205.60	250.00	44.40
July 1, 1926	10190.38	204.71	250.00	45.29
Jan. 1, 1927	10144.19	203.81	250.00	46.20
July 1, 1927	10097.07	202.88	250.00	47.12
Jan. 1, 1928	10049.01	201.94	250.00	48.06
July 1, 1928	9999.99	200.98	250.00	49.02

INVESTMENT SCHEDULE (For Exercise 1, Art. 48)

Date	Book Value	Interest on Book Value at 3%	Dividend Received	For Accumulation of Discount
July 1, 1923	9573.49			
Jan. 1, 1924	9610.69	287.20	250.00	37.20
July 1, 1924	9649.01	288.32	250.00	38.32
Jan. 1, 1925	9688.48	289.47	250.00	39.47
July 1, 1925	9729.13	290.65	250.00	40.65
Jan. 1, 1926	9771.00	291.87	250.00	41.87
July 1, 1926	9814.13	293.13	250.00	43.13
Jan. 1, 1927	9858.55	294.42	250.00	44.42
July 1, 1927	9904.31	295.76	250.00	45.76
Jan. 1, 1928	9951.44	297.13	250.00	47.13
July 1, 1928	9999.98	298.54	250.00	48.54

EXERCISES

1. Make investment schedules for Exercise 4, Art. 48, showing the amortization of the premium in the first instance, and the accumulation of the discount in the second.

2. Make an investment schedule for Exercise 6, Art. 48.

3. Make a schedule for Exercise 10, Art. 48, showing the book value of the investment at the end of each year.

50. The value of an ordinary bond when a sinking fund is created to restore the difference between the redemption value and the purchase or selling price. When a bond is purchased at a price in excess of the redemption value, the purchaser can create a sinking fund to restore the excess at the time the bond matures. When the purchase price is less than the redemption value, the seller can create a like sinking fund. When the sinking-fund rate equals the investment rate, the sinking-fund payment for the

case treated in Art. 48 is $\left(\frac{Cg}{m} - \frac{Vj}{m}\right)$ if $F < V$. Hence in this case

$$\left(\frac{Cg}{m} - \frac{Vj}{m}\right) s_{\overline{mn}|\frac{j}{m}} = V - F$$

Solving, $$V = \frac{F + \frac{Cg}{m} s_{\overline{mn}|\frac{j}{m}}}{1 + \frac{j}{m} s_{\overline{mn}|\frac{j}{m}}} = \frac{\frac{F}{s_{\overline{mn}|\frac{j}{m}}} + \frac{Cg}{m}}{\frac{1}{a_{\overline{mn}|\frac{j}{m}}}} = F\left(1 + \frac{j}{m}\right)^{-mn} + \frac{Cg}{m} a_{\overline{mn}|\frac{j}{m}}$$

In this case, as would be expected, the purchase price agrees with that in Art. 48.

When the sinking fund rate is j' converted m times per year, the sinking-fund equation becomes

$$\left(\frac{Cg}{m} - \frac{Vj}{m}\right) s_{\overline{mn}|\frac{j'}{m}} = V - F$$

$$V = \frac{F + \frac{Cg}{m} s_{\overline{mn}|\frac{j'}{m}}}{1 + \frac{j}{m} s_{\overline{mn}|\frac{j'}{m}}}$$

The same expression for V would evidently be found if $F > V$.

The discussion in this article for the value of an ordinary bond is analogous to that given in Art. 47 for an investment whose return is an annuity. A similar discussion can be given to restore the capital for each of the more general investments treated in Arts. 52 to 54 inclusive. Other illustrations of this type of investment are given in the exercises below.

EXAMPLE. A \$10000 bond issued July 1, 1923, bearing interest at 5% payable semi-annually, is to be redeemed at par in 5 years. Find its purchase price to yield 4% converted semi-annually if a sinking fund which accumulates at ($j = .035$, $m = 2$) is created to restore the premium.

SOLUTION. In this case the equation in V is

$$(250 - .02\,V)s_{\overline{10}|.0175} = V - 10000$$

Solving, $$V = \frac{10000 + 250\,s_{\overline{10}|.0175}}{1 + .02\,s_{\overline{10}|.0175}}$$

$$= \$10444.94 \quad \text{(See Example, Art. 48.)}$$

EXERCISES

1. Solve the above example if the sinking fund accumulates at (j = .04, m = 2). Ans. \$10449.13.

2. A \$5000 bond bearing interest at $5\frac{1}{2}\%$ payable semi-annually, and redeemable at par in 10 years, is bought to yield (j = .05, m = 2). If a sinking fund which accumulates at (j = .05, m = 2) is established to restore the premium, construct the sinking-fund schedule. (See Exercise 4, Art. 48.)

3. Same as Exercise 2, except that the sinking fund accumulates at (j = .045, m = 2). Ans. \$5191.89 (purchase price).

4. If the bond in Exercise 2 is bought to yield (j = .06, m = 2) and a sinking fund which accumulates at (j = .06, m = 2) is established to accumulate the discount, construct the sinking-fund schedule. (See Exercise 4, Art. 48.) Ans. \$4814.03.

5. Same as Exercise 4, except that the sinking fund accumulates at (j = .055, m = 2).

6. It is estimated that a taxicab will yield a net return of \$500 at the end of each year for 4 years, at the end of which time the taxicab will have a scrap value of \$100. If a sinking fund which accumulates at (j = .04, m = 1) is created to restore in 4 years the capital invested less the scrap value, find the purchase price to yield 10% annually. (See Exercise 2, Art. 47, and Exercise 9, Art. 48.) Ans. \$1560.55.

7. It is estimated that the income of a lumber company in developing a tract of wooded land will be \$10000 annually for 15 years, at which time production will cease and the land will be worth \$2500. If a sinking fund which accumulates at (j = .04, m = 1) is established to restore the capital invested less the value of the land, find the purchase price correct to dollars to yield 10% annually. (See Exercise 10, Art. 48.) Ans. \$67526.

51. The value of a bond purchased between interest payment dates. The expression for the value of an ordinary bond or of an annuity bond purchased, at date of issue or at an interest-bearing date, to yield a given interest rate can be used to find the value of the bond purchased at any date to yield this interest rate. If V' denotes the value of a bond n_1 years $\left(n_1 < \frac{1}{m}\right)$ after the date at which V was found in Art. 48, then, by Theorem I, Art. 15, Chapter I, it follows that $V' = V\left(1 + \frac{j}{m}\right)^{mn_1}$. In practice, however, the simple interest formula is ordinarily used for the fractional term, n_1, so that

$$V' = V(1 + jn_1)$$

Unless otherwise stated, this formula will be used in what follows to find the value of a bond purchased between interest payment dates *to yield a given interest rate.*

Bonds are often listed or quoted at a given price. The amount quoted is usually what must be paid by the purchaser for each \$100 of face value. For example, a \$10000 bond quoted at 105 can be bought for \$10500. In such cases the interest rate earned by the purchaser is not ordinarily known, so that the preceding formula cannot be applied. When a bond quoted at a certain price is sold between interest payment dates, the seller is entitled to part of the interest due on the bond at the next interest payment date, since he has held the bond during a part of this period. In practice, the part he gets is usually the simple interest on the face of the bond at the rate named in the bond for the part of the period (n_1 years) during which he holds it. The value by this method is said to be the *value at a certain price and accrued interest.* In the case of an ordinary bond, as treated in Art. 48, sold at a price P and accrued interest this value is given by

$$V' = P + Cgn_1$$

EXAMPLE 1. Find the value of the bond in the example in Art. 48 to yield 4% semi-annually, if purchased October 1, 1925.

SOLUTION. By Art. 48, or by the first schedule in Art. 49, the value of this bond July 1, 1925, is \$10280.07. Hence the purchase price V' to yield ($j = .04$, $m = 2$) is given by

$$\begin{aligned} V' &= 10280.07\ (1.01) \\ &= \$10382.87 \end{aligned}$$

EXAMPLE 2. If the bond in the example in Art. 48 were quoted at 102 on October 1, 1925, find its purchase price with accrued interest.

SOLUTION. The price at which it is quoted is \$10200, and the accrued interest on the bond is \$125. Hence the purchase price with accrued interest, V', is given by

$$\begin{aligned} V' &= 10200 + 125 \\ &= \$10325 \end{aligned}$$

EXERCISES

1. A \$100 bond dated October 15, 1920, bearing interest at 6% payable annually, and maturing in 10 years, was purchased January 1, 1924 to yield ($j = .05$, $m = 2$). Find the purchase price. Ans. \$106.52.

2. Same as Exercise 1, except that the bond is redeemable at 102.

3. A \$500 bond bearing interest at 5% semi-annually, payable June 15 and December 15, was purchased on August 1 at $98\frac{7}{8}$ and accrued interest. Find the purchase price. Ans. \$497.57.

4. A \$500 Liberty Bond was purchased at 99−26. If the bond bears interest at $4\frac{1}{4}$% payable semi-annually, find the purchase price 23 days before an interest date; 23 days after an interest date. Ans. \$508.33; \$500.42.

5. A \$1000 bond bearing interest at $5\frac{1}{2}$% payable semi-annually was purchased at $103\frac{1}{4}$, 95 days after an interest date. Find the purchase price.

52. The value of serial bonds or bonds to be redeemed in equal periodic instalments. The return is that of a set of ordinary bonds having equal face values. In an ordinary bond the face value is paid in a single instalment. In annuity bonds (Art. 46) the face value is paid in instalments at the interest payment dates such that the instalment plus the interest is constant or nearly constant. In *serial bonds* the face value is paid in instalments which are equal or nearly equal. The values of all bonds whose face is paid in known instalments may evidently be found by summing the values of the ordinary bonds composing them. In some important cases, however, simpler methods may be employed for computing the values of such bonds. In this article methods for finding the values of serial bonds whose face value is paid in equal periodic instalments are treated. The treatment is given first for a special case and then for the general case.

Special Case: *The value of serial bonds whose face value is paid in equal instalments at the interest payment dates.* The return in this case consists of the annuity composed of the instalments on the bonds and the decreasing annuity composed of the interest payments on the instalments. If C_1 denoted the amount of each instalment the present value of the annuity composed of them is $C_1 a_{\overline{mn}|\frac{j}{m}}$, and the present value of the decreasing annuity composed of the interest payments is, by Formula (8), Art. 32, Chapter II, $\frac{C_1 g}{m}\left(\frac{mn - a_{\overline{mn}|\frac{j}{m}}}{\frac{j}{m}}\right)$. Hence

$$V = C_1 a_{\overline{mn}|\frac{j}{m}} + \frac{C_1 g}{m}\left(\frac{mn - a_{\overline{mn}|\frac{j}{m}}}{\frac{j}{m}}\right)$$

This formula can be written in two other forms, analogous to those for an ordinary bond (Art. 48). These are

$$V = mnC_1 + C_1\left(\frac{g}{m} - \frac{j}{m}\right)\left(\frac{mn - a_{\overline{mn}|\frac{j}{m}}}{\frac{j}{m}}\right)$$

$$V = mn\frac{C_1 g}{j} + \left(C_1 - \frac{C_1 g}{j}\right) a_{\overline{mn}|\frac{j}{m}}$$

The right member of the first involves the values of an annuity, and of a decreasing annuity; that of the second involves the value of a decreasing annuity; and that of the third, the value of an annuity. In finding the values of serial bonds of this type the principles underlying the formulas should be used rather than the formulas. These principles are seen at once by expressing the formulas verbally. For example, by the last formula the value of the bonds equals the amount of mn investments of $\frac{C_1 g}{j}$ each plus or minus the present value of an annuity whose rent is the difference between an instalment on the bond and the amount of one of these investments. These three principles will now be applied in turn to solve

EXAMPLE 1. Serial bonds to the amount of $10000 bearing interest at 5% payable semi-annually are to be redeemed in 5 years in ten equal semi-annual instalments of $1000 each. Find the value of these bonds to yield the purchaser 4% converted semi-annually.

SOLUTION 1. The present value of the instalments is $1000\, a_{\overline{10}|.02}$, and that of the decreasing annuity composed of the interest payments is $25\,\frac{10 - a_{\overline{10}|.02}}{.02}$. Hence

$$\begin{aligned} V &= 1000\, a_{\overline{10}|.02} + 25\,\frac{10 - a_{\overline{10}|.02}}{.02} \\ &= \$10254.354 \end{aligned}$$

SOLUTION 2. If ten $1000 investments are made at 4% payable semi-annually for terms ranging by half years, from ½ to 5 years, the return on these investments differs from the return on the bonds by the decreasing annuity which is the difference between the interest payments on the bonds and those on the investments. The rent payments of this decreasing annuity begin with $50 and decrease $5 each period. The present value of the decreasing

annuity is then $5\,\frac{10 - a_{\overline{10|}.02}}{.02}$. Hence

$$V = 10000 + 5\,\frac{10 - a_{\overline{10|}.02}}{.02}$$
$$= \$10254.354$$

SOLUTION 3. If ten \$1250 investments are made at 4% payable semi-annually for terms ranging by half years, from ½ to 5 years, the return on these investments differs from the return on the bonds by an annuity which is the difference between the instalments on the bonds and those of the investments. The rent of this annuity is 250 and its present value is $250\,a_{\overline{10|}.02}$. Hence

$$V = 12500 - 250\,a_{\overline{10|}.02}$$
$$= \$10254.354$$

EXERCISE. Find the value of the bonds in this example to yield 6% converted semi-annually. Ans. \$9755.034.

GENERAL CASE: *The value of serial bonds whose face is paid in equal periodic instalments at periods of r years, the first instalment being paid in n_1 years.* The number of instalments is $\frac{n-(n_1-r)}{r}$. When $n_1 = r = \frac{1}{m}$ this case reduces to the special case treated above. Each of the principles used in the special case may be applied to this general case. If the first two are used, the decreasing annuities must be replaced by sets of annuities of the type treated in Exercise 3, Art. 32, Chapter II. The third principle may be applied, however, without change. In this method, corresponding to each instalment on the bond an investment of $\frac{C_1 g}{j}$ is made for a term equal to that of the instalment. These investments yield the interest payments which the bonds yield, so that the difference between the return on the bonds and that on the investments is an annuity whose rent is $C_1 - \frac{C_1 g}{j}$, of rent period r, having the first payment in n_1 years and the last in n years. The present value of this annuity, whose term begins in $n_1 - r$ years, is $\left(C_1 - \frac{C_1 g}{j}\right)\frac{a_{\overline{mn|}\frac{j}{m}} - a_{\overline{m(n_1-r)|}\frac{j}{m}}}{s_{\overline{mr|}\frac{j}{m}}}$. Hence

$$V = \frac{n-(n_1-r)}{r}\,\frac{C_1 g}{j} + \left(C_1 - \frac{C_1 g}{j}\right)\frac{a_{\overline{mn|}\frac{j}{m}} - a_{\overline{m(n_1-r)|}\frac{j}{m}}}{s_{\overline{mr|}\frac{j}{m}}}$$

This method is illustrated by the solution of

EXAMPLE 2. A \$10000 loan bearing 5% payable semi-annually is to be repaid in ten \$1000 instalments, the first in 5 years and the others at periodic intervals of 2 years. Find the purchase price to yield 4% converted semi-annually.

SOLUTION. If ten investments of 1250 each are made at 4% payable semi-annually for terms of 5, 7, 9, . . . , 23 years respectively, the interest payments on them are the same as those on the loan. The return on these ten investments exceeds that on the loan by an annuity whose rent is \$250, rent period 2 years, first payment in 5 years and last in 23 years. The value of this deferred annuity (deferred 3 years) is $250 \dfrac{a_{\overline{46}|.02} - a_{\overline{6}|.02}}{s_{\overline{4}|.02}}$. Hence

$$V = 12500 - 250 \frac{a_{\overline{46}|.02} - a_{\overline{6}|.02}}{s_{\overline{4}|.02}}$$
$$= \$11026.61$$

EXERCISE. Find the purchase price of the loan to yield 6% converted semi-annually.

The same method applies to the relation of serial bonds which are redeemed above par.

EXERCISES

1. Solve Example 2 above by use of the result found in Exercise 3, Art. 32.

2. Solve Example 2 above by summing the values of the annuity consisting of the ten instalments and of the ten annuities consisting of the interest payments on these instalments.

3. Serial bonds to the amount of \$10000 are redeemable in 10 equal annual instalments. If the bonds bear interest at 6% payable annually, find the purchase price at date of issue to yield ($j = .065$, $m = 1$). Ans. \$9783.76.

4. Serial bonds to the amount of \$25000 are redeemable at 102 in ten equal semi-annual instalments. If the bonds bear interest at $5\frac{1}{2}$% payable semi-annually, find the purchase price at date of issue to yield ($j = .06$, $m = 2$). Ans. \$25120.30.

5. Serial bonds to the amount of \$25000 are redeemable at 105 in five equal annual instalments, the first redemption to take place 3 years from date of issue. If the bonds bear interest at 6% payable annually, find the purchase price at date of issue to yield ($j = .055$, $m = 1$). Ans. \$26487.96.

6. Same as Exercise 3, except that the bonds are redeemable in 5 equal biennial instalments, the first redemption to take place 3 years from date of issue.

53. Investment schedules for instalment bonds. An investment schedule for instalment bonds differs from a schedule for an ordinary bond only in that it provides a column for the redemption payments.

EXAMPLE. $10000 of serial bonds issued January 1, 1919, bearing interest at 5% payable semi-annually, are redeemed in ten equal semi-annual instalments of $1000 each. If these bonds are bought to yield the purchaser 6% converted semi-annually, construct an investment schedule.

SOLUTION. By the first exercise in Art. 52, the purchase price of these bonds is $9755.034.

INVESTMENT SCHEDULE

DATE	BOOK VALUE	INTEREST ON BOOK VALUE AT 3%	DIVIDEND RECEIVED	FOR ACCUMULATION OF DISCOUNT	REDEMPTION PAYMENTS
Jan. 1, 1919	9755.034				
July 1, 1919	8797.685	292.651	250	42.657	1000
Jan. 1, 1920	7836.616	263.931	225	38.931	1000
July 1, 1920	6871.714	235.098	200	35.098	1000
Jan. 1, 1921	5902.865	206.151	175	31.151	1000
July 1, 1921	4929.951	177.086	150	27.086	1000
Jan. 1, 1922	3952.850	147.899	125	22.899	1000
July 1, 1922	2971.436	118.586	100	18.586	1000
Jan. 1, 1923	1985.579	89.143	75	14.143	1000
July 1, 1923	995.146	59.567	50	9.567	1000
Jan. 1, 1924	.000	29.854	25	4.854	1000

EXERCISES

1. Construct an investment schedule for Exercise 3, Art. 52.

2. Construct an investment schedule for Exercise 4, Art. 52.

3. Construct an investment schedule for Exercise 6, Art. 52.

54. The value of any investment whose return is known. When the return on an investment is different from the types considered in the preceding articles, it can always be resolved into sets which belong to these types. It is evident, for example, that any return is a set of single sums. Again the return on any instalment bond is composed of sets of returns on ordinary bonds. Often a given return can be resolved into sets in two or more ways whose values can be computed with about equal ease. In finding the value of any investment whose return is known the aim should be to resolve the return into sets such that the computations needed in finding their values can be performed with minimum effort. Some illustrations are afforded by the solution of the

Example. A debt of \$10000 consisting of 100 bonds of \$100 denomination bearing interest at 5% payable semi-annually is to be paid, principal and interest, in ten semi-annual instalments as nearly equal as possible. Find the purchase price of these bonds to yield the investor 4% converted semi-annually.

The retirement of the bonds in accordance with the solution of the example in Art. 36 will be used as a basis for finding the purchase price.

Solution 1. This solution is based on resolving the return into three sets of the type treated in Art. 53. The first set consists of the first three instalments of \$900 each and their interest payments; the second consists of the next four instalments of \$1000 each and their interest payments; the third consists of the last three instalments of \$1100 each and their interest payments. By Art. 52 the respective values of these sets are:

$$3(1125) - 225\, a_{\overline{3}|.02}$$
$$4(1250) - 250\,(a_{\overline{7}|.02} - a_{\overline{3}|.02})$$
$$3(1375) - 275\,(a_{\overline{10}|.02} - a_{\overline{7}|.02})$$

Adding these gives

$$V = 12500 - 275\, a_{\overline{10}|.02} + 25\, a_{\overline{7}|.02} + 25\, a_{\overline{3}|.02}$$
$$= \$10263.69$$

Solution 2. This solution is based on resolving the return into two sets; one consists of the ten instalments on the face of the bonds and the other of the interest payments. The value of the first set can be easily found by resolving it into an annuity having ten rent payments of \$1100 each less an annuity having seven rent payments of \$100 each, less again an annuity having three rent payments of \$100 each. The value of the second set can be found by resolving it into three decreasing annuities analogous to the three annuities into which the first set is resolved. Hence

$$V = 1100\, a_{\overline{10}|.02} - 100\, a_{\overline{7}|.02} - 100\, a_{\overline{3}|.02} + 27.50 \left(\frac{10 - a_{\overline{10}|.02}}{.02}\right)$$
$$- 2.50 \left(\frac{7 - a_{\overline{7}|.02}}{.02}\right) - 2.50 \left(\frac{3 - a_{\overline{3}|.02}}{.02}\right)$$
$$= 12500 - 275\, a_{\overline{10}|.02} + 25\, a_{\overline{7}|.02} + 25\, a_{\overline{3}|.02}$$
$$= \$10263.69$$

Solution 3. This solution is based on resolving the return into two sets; one consists of the ten instalments and their interest payments at 4% payable semi-annually, the other consists of the difference between the interest payments on the bond and the interest payments in the first set. The value of the first set is 10000; that of the second can be found by resolving it into three decreasing annuities as in solution 2. Hence

$$V = 10000 + 5.50 \left(\frac{10 - a_{\overline{10}|.02}}{.02}\right) - .50 \left(\frac{7 - a_{\overline{7}|.02}}{.02}\right) - .50 \left(\frac{3 - a_{\overline{3}|.02}}{.02}\right)$$
$$= 12500 - 275\, a_{\overline{10}|.02} + 25\, a_{\overline{7}|.02} + 25\, a_{\overline{3}|.02}$$
$$= \$10263.69.$$

EXERCISES

1. Construct an investment schedule for the above example.

2. Solve this example if the investment rate is 6% converted semi-annually. Ans. \$9746.22.

3. In Exercise 1, Art. 36, find the purchase price of the bonds at date of issue to yield ($j = .06$, $m = 2$).

4. In Exercise 2, Art. 36, find the purchase price of the bonds at date of issue to yield ($j = .045$, $m = 2$).

5. A corporation issues bonds to the amount of \$600000 bearing interest at 5% payable semi-annually. The bonds are redeemable as follows: \$100000 in 5 years at 100; \$200000 in 8 years at 102; \$300000 in 11 years at 105. Find the purchase price at date of issue to yield ($j = .055$, $m = 2$). Construct an investment schedule. Ans. \$590029.87.

55. The number of equal periodic payments needed to purchase an investment of known value when the interest rate is given. Building and Loan Association Stocks. In the investments treated in the preceding articles, the purchase price is determined in a lump sum at the time the investment is made. Some investments, however, are paid for on the instalment plan. A good illustration is afforded by stocks issued by some building and loan associations. In such cases the payments are usually equal and periodic and

they are accumulated at a specified rate; the problem is to find the number of payments. When the payments accumulate to the value of the stock, the stock is said to *mature*. Stock of given value purchased by a set of equal periodic payments will usually mature at a time within the period just after the last full payment is made. The subscriber to the stock is ordinarily notified, however, of its maturity at the end of the next period at which time settlement is made.

EXAMPLE. A member of a building and loan association pays $1.00 at the beginning of each month on a $100 share of stock. Find the number of payments and time of maturity if the association pays 6% converted monthly.

SOLUTION. The payments constitute an annuity due of term n years. Equating the value of this annuity due just after the last payment is made to 100 gives:

$$\frac{1.005^{12n} - 1}{.005} = s_{\overline{12n}|.005} = 100$$

By Table V, $s_{\overline{81}|.005} = 99.56$, and $s_{\overline{82}|.005} = 101.06$.

Hence 81 payments are needed and the stock matures in 6 years, 8 months, and d days. To find d the ordinary simple interest formula may be used; $100 - 99.56 = 0.44$ is the interest on 99.56 for d days. This gives

$$.44 = 99.56 \frac{d}{360} (.06)$$

Solving, $$d = 27$$

Hence 81 payments mature the stock in 6 years, 8 months, and 27 days. In this case settlement would usually be made at the end of 6 years and 9 months, at which time the 81 payments made have a value of 99.56 $(1 + .005) = 101.06$. Settlement at this time requires the delivery of the stock and the cash payment of 6 cents to the subscriber.

EXERCISES

1. A member of a building and loan association pays $1 at the beginning of each month on a $100 share of stock. If the association pays 5% converted monthly show that the stock will mature in 6 years and 11 months, the final payment being 67 cents.

2. Find the book values of the payments on the stock in Exercise 1 at the end of 4 years; at the end of 5 years and 6 months. Ans. $54.24; $77.10.

56. To find the rate in investment, debt, and other problems in finance. Interest and discount rates are frequently the unknowns in investment, debt, and other problems in finance. A bond purchased at a quoted value, a debt amortized by a known

set of payments, a depreciation estimated by the use of the simple discount formula, are illustrations. The equation determining the rate in any rate problem is found, in the usual manner, by equating the values of equivalent sets of sums. Methods for determining rates based on the compound interest formula and on annuity formulas have been presented in Art. 12, Chapter I, and in Art. 26, Chapter II. By use of one or more of these methods any rate equation arising in elementary finance can be solved. In the next four articles solutions of some important types of rate equations are given; the types treated are introduced in the order of their difficulty.

57. Rate equations based on the value of one sum. Various problems involving the determination of rates based on the interest and discount formulas for the value of one sum have been given in Chapter I. Two problems which relate to depreciation are solved in this article.

EXAMPLE 1. An article costing \$1000 has a value of \$100 at the end of 10 years. If the simple discount formula is used in estimating depreciation, find the discount rate.

SOLUTION. By the simple discount formula, $D = Snd$,

$$900 = 1000(10)d$$

Solving,

$$d = .09$$

EXAMPLE 2. An article costing \$1000 has a value of \$100 at the end of 10 years. If the compound discount formula, with annual conversions, is used in estimating depreciation, find the discount rate.

SOLUTION. By the compound discount formula, $P = S\left(1 - \frac{f}{m}\right)^{mn}$

$$100 = 1000(1 - f)^{10}$$

Solving,

$$1 - f = \sqrt[10]{.1}$$

$$f = 1 - \sqrt[10]{.1} = .206$$

58. Rate equations based on the value of an annuity. Two examples of rate equations based on the value of an annuity are given in Art. 26, Chapter II. In this article two additional examples are solved.

EXAMPLE 1. A debt of \$10000 is to be amortized by payments of \$1150 at the end of each half year for 5 years. If $m = 2$, find $\frac{j}{2}$.

Equating the present value of the payments to 10000 gives

$$1150\frac{1-\left(1+\frac{j}{2}\right)^{-10}}{\frac{j}{2}}=10000$$

or

$$\frac{1-\left(1+\frac{j}{2}\right)^{-10}}{\frac{j}{2}}=8.69565217$$

Solution by Interpolation. In this case,

$$a_{\overline{10}|\frac{j}{2}}=\frac{1-\left(1+\frac{j}{2}\right)^{-10}}{\frac{j}{2}}$$

can be used to construct a table of values for interpolation. By Table VI $a_{\overline{10}|.025}=8.75206393$, and $a_{\overline{10}|.0275}=8.64007616$. Hence $\frac{j}{2}$ lies between .025 and .0275. Interpolating from the table

$\frac{j}{2}$	$a_{\overline{10}\vert\frac{j}{2}}$
.025	8.7521
$\frac{j}{2}$	8.6957
.0275	8.6401

gives $\dfrac{\frac{j}{2}-.025}{.0025}=\dfrac{.0564}{.1120}$

Solving, $\frac{j}{2}=.02626$

By the use of a seven-place table of logarithms it is found that $a_{\overline{10}|.02625}=8.6958$, and $a_{\overline{10}|.02626}=8.6953$. Hence $\frac{j}{2}$ lies between .02625 and .02626.

Exercise 1. Solve this example by using Table VII for interpolation. $\left(\text{In this example, } \frac{1}{a_{\overline{10}|\frac{j}{2}}}=.11500000.\right)$

Exercise 2. Solve this example by Newton's method, using $\frac{j}{2}=.0262+h$.

Exercise 3. Solve this example by the method of iteration.

Example 2. A debt of \$10000 is to be amortized by payments of \$1150 at the end of each half year for 5 years. Find the effective interest rate i.

Equating the present value of the payments to \$10000 gives

$$1150\frac{1-(1+i)^{-5}}{(1+i)^{\frac{1}{2}}-1}=10000$$

or

$$\frac{1-(1+i)^{-5}}{(1+i)^{\frac{1}{2}}-1}=8.69565217$$

SOLUTION 1. In this solution $V = \dfrac{1-(1+i)^{-5}}{(1+i)^{\frac{1}{2}}-1}$ is used to construct a table of values for interpolation.

i	V
.05	8.7659
i	8.6957
.055	8.6564

$$\frac{i-.05}{.005} = \frac{.0702}{.1095}$$

Solving, $i = .0532$

SOLUTION 2. In this solution, use is made of the formula $1+i = \left(1+\frac{j}{2}\right)^2$ which connects the corresponding rates i and $j_{(2)}$. Replacing $1+i$ by $\left(1+\frac{j}{2}\right)^2$, the equation to be solved becomes

$$\frac{1-\left(1+\frac{j}{2}\right)^{-10}}{\frac{j}{2}} = 8.69565217$$

By solution of Example 1, $\frac{j}{2} = .02625$ or $.02626$. Substituting either of these values for $\frac{j}{2}$ into the relation $1+i = \left(1+\frac{j}{2}\right)^2$ gives $i = .0532$.

EXERCISES

1. A lease whose return is \$2500 at the end of each six months is sold for \$70000, 30 years before it expires. If $m = 2$, find the rate of interest on the investment. Ans. .0589.

2. A house valued at \$6800 is purchased for \$1000 in cash and \$1000 at the end of each year for 8 years. What interest rate converted annually will make the payments equivalent to the value of the house. Ans. .0775.

3. A piano valued at \$500 is purchased for \$100 in cash and \$35 at the end of each month for 12 months. Find the interest rate, converted monthly, which will make the present value of the payments equal to the value of the piano; find also the simple discount rate. Ans. $\frac{j}{12} = .0076$; $d = .0879$.

4. At the time of maturity of a certain life insurance policy the beneficiary was offered \$10000 in cash, or \$1161.75 in cash, and a like amount at the end of each year for 9 years. If $m = 1$, find the interest rate used.

5. A deposit of \$10 at the end of each month for 10 years was made in a saving's bank. At the end of this time there was a credit of \$1548.57 to this account. If $m = 2$, find the interest rate used. Ans. .0500.

59. Rate equations based on the value of an ordinary bond.

Example. A \$10000 bond bearing interest at 5% payable semi-annually is to be redeemed at par in 5 years. This bond is purchased for \$10200. If $m = 2$, find $\frac{j}{2}$.

Equating the value of the bond at the rate j payable semi-annually to 10200 gives

$$10200 = 10000\left(1 + \frac{j}{2}\right)^{-10} + 250\,\frac{1 - \left(1 + \frac{j}{2}\right)^{-10}}{\frac{j}{2}}$$

Solution by interpolation. In this case

$$V = 10000\left(1 + \frac{j}{2}\right)^{-10} + 250\,\frac{1 - \left(1 + \frac{j}{2}\right)^{-10}}{\frac{j}{2}}$$

can be used to construct a table of values of V and $\frac{j}{2}$ for interpolation. The upper set of values in the following table are readily found by the use of Tables IV and VI; the lower set is evident:

$\frac{j}{2}$	V
.0225	10221.6554
.02274	10200.12
$\frac{j}{2}$	10200.00
.02275	10199.23
.025	10000.00

Interpolating between the upper and the lower sets gives

$$\frac{\frac{j}{2} - .0225}{.0025} = \frac{21.6554}{221.6554}$$

$$\frac{j}{2} = .02274$$

The two inner sets can be computed by the use of a seven-place table of logarithms. This table shows that $\frac{j}{2}$ lies between .02274 and .02275.

EXERCISES

1. A \$100 bond bearing interest at $5\frac{1}{2}\%$ payable semi-annually and redeemable at par in 10 years is purchased at date of issue for \$98.50. What rate converted semi-annually does the investment yield? Ans. .0570.

2. Same as Exercise 1 except that the bond is redeemable at 102.

Ans. .0585.

3. A \$1000 bond issued by a corporation bears interest at $5\frac{1}{4}\%$ payable semi-annually and is redeemable at par in 10 years. A \$1000 bond issued by a second corporation bears interest at 5% payable semi-annually and is

redeemable at 102 in 10 years. Which of the two bonds is the better investment if the first is quoted at 100½ and the second at 99½?

4. A $100 liberty bond bearing interest at 4¼% payable semi-annually and redeemable at par in 5 years was purchased for 96−21. What rate converted semi-annually does the investment yield? Ans. .0501.

60. Rate equations based on the value of any set of sums. The rate equations which occur in elementary finance are usually of the types treated in the three preceding articles. A rate equation based on the value of any set of sums can be solved, however, by the use of the methods employed in these articles.

Example. A share of common stock bought at par ($100) yielded dividends at 6% payable quarterly, and a bonus of 2% at the end of each year, for 10 years, when it was sold at $105. If $m = 4$, find $\frac{j}{4}$.

Equating the present value of the return on the stock to $100 gives

$$100 = \frac{3}{2}\,\frac{1-\left(1+\frac{j}{4}\right)^{-40}}{\frac{j}{4}} + 2\,\frac{1-\left(1+\frac{j}{4}\right)^{-40}}{\left(1+\frac{j}{4}\right)^{4}-1} + 105\left(1+\frac{j}{4}\right)^{-40}$$

Solution. In this case

$$V = \frac{3}{2}\,\frac{1-\left(1+\frac{j}{4}\right)^{-40}}{\frac{j}{4}} + 2\,\frac{1-\left(1+\frac{j}{4}\right)^{-40}}{\left(1+\frac{j}{4}\right)^{4}-1} + 105\left(1+\frac{j}{4}\right)^{-40}$$

can be used to construct a table of values for V and $\frac{j}{4}$ for interpolation,

$\frac{j}{4}$	V
.02	101.8609
$\frac{j}{4}$	100.0000
.0225	95.0710

$$\frac{\frac{j}{4} - .02}{.0025} = \frac{1.8609}{6.7899}$$

$$\frac{j}{4} = .0207$$

EXERCISES

1. A debt of $5000 was paid by payments of $1000 at the end of each year for 3 years, and $1600 at the end of each of the two following years. What rate of interest converted annually was used? Ans. .0690.

2. Common stock of a corporation bought at par paid dividends at 6% semi-annually for 3 years, 8% semi-annually for 2 years, and 7% semi-annually

for 2 years. At the end of the 7 years the stock was sold at 102. What rate of interest converted semi-annually was received on the investment?

Ans. .0701.

3. Five shares of common stock in a mercantile company were purchased for $500. At the end of each six months for two years the dividend on the stock was $20.00. For the following two years no dividends were paid. For the following six years, a dividend of $15.00 was paid at the end of each six months. At the end of the 10 years the stock was sold for $490. What rate of interest converted semi-annually was received on the investment?

Ans. .0501.

4. A $100 share of a building and loan association stock bought on monthly payments matures at the time of the 80th payment. If the 80th payment is 18 cents and each of the others is $1.00, find $\frac{j}{2}$. Ans. .0344.

5. The serial bonds in Example 1, Art. 52, were purchased for $10125. If $m = 2$, find $\frac{j}{2}$. Ans. .0450.

DEPRECIATION AND CAPITALIZED COST

61. Methods of estimating depreciation charges. Definitions. Buildings, machinery, and other property used in business enterprises deteriorate in value. Part of such losses in value can be provided for by current repairs. The loss in value that cannot be provided for by current repairs is called *depreciation.* When an article has depreciated to an extent that makes replacement necessary the total loss in value due to depreciation is the difference between its cost and its *salvage* or *scrap* value at the time of replacement. Here, as elsewhere in business, capital invested should not be impaired. To provide for depreciation losses a common practice is to set aside sums, called *depreciation charges,* at periodic intervals. The *book value* of an article at any time is its cost less the value at the time, of the depreciation charges. The *wearing value* of an article at any time is its book value at the time less the salvage value. The *total wearing value* is its cost less its salvage value. The *condition per cent* of an article at any time is its wearing value divided by its cost.

All articles do not depreciate in the same manner, so that different methods of estimating depreciation charges are necessary. Practically all methods of treating depreciation are alike in that they aim to restore the total wearing value of an article at the end

of its depreciation term; that is, at the time replacement is necessary. In Arts. 62 to 65 inclusive, four different methods of estimating depreciation charges are presented. In the examples and exercises in these articles n denotes the number of years in the depreciation term, and t denotes the number of years from the beginning to any date within the term.

62. The straight-line or simple-discount method. By this method the simple discount formulas $D = Snd$, $P = S(1 - nd)$, are used in estimating the depreciation and the book value at any time. It is called the straight-line method because the graphs of these formulas for given values of S and d are straight lines (Art. 7, Chapter I). The simple discount rate in a given problem can be found from $D = Snd$ by replacing S by the cost of the article, D by the total depreciation, and n by the depreciation term. The annual depreciation is the cost times the simple discount rate.

EXAMPLE. An article costing $1000 has a salvage value of $100 at the end of 14 years. Use the straight-line method to find formulas for the depreciation D and the book value B at the end of t years, and construct a depreciation schedule.

SOLUTION. Substituting $D = 900$, $S = 1000$, and $n = 14$ into $D = Snd$, gives $d = \frac{9}{140} = .06429$. Replacing S by 1000, P by B, d by $\frac{9}{140}$, and n by t in the simple discount formulas gives

$$D = \frac{450\,t}{7}$$

$$B = 1000 - \frac{450\,t}{7}$$

By use of these formulas B and D can be computed for assigned values of t. The annual depreciation charge is $\frac{450}{7}$; it is also the difference between consecutive values of B or consecutive values of D when t is assigned integral values. These results are shown in the depreciation schedule on the next page.

DEPRECIATION SCHEDULE

Year	Book Value at End of Year	Depreciation Charge at End of Year	Total Depreciation Charge at End of Year
0	1000.00		
1	935.71	64.29	64.29
2	871.43	64.28	128.57
3	807.14	64.29	192.86
4	742.86	64.28	257.14
5	678.57	64.29	321.43
6	614.28	64.29	385.72
7	550.00	64.28	450.00
8	485.71	64.29	514.29
9	421.43	64.28	578.57
10	357.14	64.29	642.86
11	292.85	64.29	707.15
12	228.57	64.28	771.43
13	164.28	64.29	835.72
14	100.00	64.28	900.00

Exercise 1. If C denotes the cost, S the scrap value, and n the depreciation term of an article, show that the straight-line method gives the formulas

$$D = \frac{C - S}{n}t, \qquad B = C - \frac{C - S}{n}t$$

Exercise 2. Apply the formulas in Exercise 1 to the above example.

EXERCISES

1. A machine costing \$2500 is depreciated 2% monthly by the straight-line method. If the machine has a scrap value of \$200, construct the depreciation schedule for the life of the machine.

2. The owner of an automobile in Ohio is required to list his automobile for taxation in accordance with the following schedule: first year, at 70% of list

price; second year, 60%; third year, 50%; fourth year, 40%; fifth year, 30%; sixth year, 20%; after the sixth year, at its actual value in money. Construct a depreciation schedule for the first six years for an automobile bought in 1916 at a cost of \$3000 assuming that the list price remains constant.

3. If the rate of depreciation d is given, show that the straight-line method is not applicable if $n > \frac{1}{d}$.

63. The constant percentage or compound discount method. By this method the compound discount formulas, $P = S\left(1 - \frac{f}{m}\right)^{mn}$, $D = S - P$, are used in estimating the book value and the depreciation at any time. It is called the constant percentage method since the annual depreciation charge for each period is a constant percentage of the book value at the beginning of the period. The constant percentage or compound discount rate in a given problem can be found from $P = S\left(1 - \frac{f}{m}\right)^{mn}$ by replacing S by the cost of the article, P by the salvage value, n by the depreciation term, and m by the number of discount conversions per year.

Example. An article costing \$1000 has a salvage value of \$100 at the end of 14 years. Use the constant percentage method to find formulas for the book value B and the total depreciation D at the end of t years, and construct a depreciation schedule.

Solution. Substituting $P = 100$, $S = 1000$, $n = 14$, and $m = 1$ into $P = S\left(1 - \frac{f}{m}\right)^{mn}$ gives $1 - f = \sqrt[14]{.1} = .84834$, or $f = .15166$. Replacing S by 1000, P by B, $1 - f$ by $(.1)^{\frac{1}{14}}$, n by t, and m by 1 in the compound discount formulas gives

$$B = 1000(.1)^{\frac{t}{14}}$$

$$D = 1000 - 1000(.1)^{\frac{t}{14}}$$

By use of these formulas and a table of logarithms, B and D can be computed readily for assigned values of t. The annual depreciation charges can then be computed by finding the differences between consecutive book values corresponding to integral values of t; they can also be found by multiplying each book value by f. The results are shown in the depreciation schedule on the next page.

DEPRECIATION SCHEDULE

Year	Logarithm of Book Value	Book Values at End of Year	Depreciation Charge at End of Year	Total Depreciation Charge at End of Year
0	3.0000000	1000.00		
1	2.9285714	848.34	151.66	151.66
2	2.8571428	719.68	128.66	280.32
3	2.7857142	610.54	109.14	389.46
4	2.7142856	517.95	92.59	482.05
5	2.6428570	439.40	78.55	560.60
6	2.5714284	372.76	66.64	627.24
7	2.4999998	316.23	56.53	683.77
8	2.4285712	268.27	47.96	731.73
9	2.3571426	227.58	40.69	772.42
10	2.2857140	193.07	34.51	806.93
11	2.2142854	163.79	29.28	836.21
12	2.1428568	138.95	24.84	861.05
13	2.0714282	117.87	21.08	882.13
14	1.9999996	100.00	17.87	900.00

The first logarithm in the second column is $\log [1000\,(.1)^{\frac{1}{14}}]$; each of the other logarithms is obtained by adding $\log (.1)^{\frac{1}{14}} = \bar{1}.9285714$ to the one just preceding it.

Exercise 1. Compute the book values in the above schedule arithmetically by multiplying in turn by the value of $(1 - f)^{\frac{1}{14}}$.

Exercise 2. If C denotes the cost, S the scrap value, n the depreciation term of an article and m equals unity, show that the constant percentage method gives the formulas

$$B = C\left(\frac{S}{C}\right)^{\frac{t}{n}}, \qquad D = C - C\left(\frac{S}{C}\right)^{\frac{t}{n}}$$

Exercise 3. Apply the formulas in Exercise 2 to the above example.

EXERCISES

1. A home costing \$6000 is depreciated at 5% annually by the constant percentage method. Find its book value at the end of the eighth year.

Ans. \$3980.52.

2. If a house costing \$10000 depreciates to \$7000 in eight years, find the constant annual rate of depreciation to four places of decimals. Ans. .0436.

3. A machine costing \$2500 is depreciated 2% monthly by the constant percentage method. If the machine has a scrap value of \$200 at the end of 46 months, construct the depreciation schedule for the life of the machine.

4. By the constant percentage method an article costing \$1000 depreciates to \$598.75 in n years at 5% converted annually. Find n. Ans. 10.

5. Show that the constant percentage method is not applicable when $S = 0$, S being the salvage value.

64. The sinking-fund method. By this method the annual depreciation charge equals the rent of an annuity whose amount at the end of the depreciation term, at a given interest rate, equals the cost of the article less its scrap value.

Example. An article costing \$1000 has a scrap value of \$100 at the end of 14 years. Use the sinking-fund method, at 5% interest converted annually, to find formulas for the book value and the depreciation charge at the end of t years, and construct a depreciation schedule.

Solution. The annual sinking-fund payment is $900 \dfrac{1}{s_{\overline{14}|.05}} = 45.9216$. The formulas for D and B at the end of t years are then

$$D = 45.9216\, s_{\overline{t}|.05}$$

$$B = 1000 - 45.9216\, s_{\overline{t}|.05}$$

By use of these formulas, B and D can be computed for assigned values of t. B and D can also be computed by constructing a schedule analogous to that in Art. 43. The results are shown in the depreciation schedule on the next page.

DEPRECIATION SCHEDULE

Year	Book Value at End of Year	Payment to Sinking Fund at End of Year	Interest Due on Fund at End of Year	Total in Fund at End of Year
0	1000.00			
1	954.08	45.92		45.92
2	905.86	45.92	2.30	94.14
3	855.23	45.92	4.71	144.77
4	802.07	45.92	7.24	197.93
5	746.25	45.92	9.90	253.75
6	687.64	45.92	12.69	312.36
7	626.10	45.92	15.62	373.90
8	561.49	45.92	18.69	438.51
9	493.64	45.92	21.93	506.36
10	422.40	45.92	25.32	577.60
11	347.60	45.92	28.88	652.40
12	269.06	45.92	32.62	730.94
13	186.59	45.92	36.55	813.41
14	100.00	45.92	40.67	900.00

Exercise 1. Use the above formulas to compute the values of B and D at the end of 6 years.

Exercise 2. If C denotes the cost, S the scrap value, and n the depreciation term of an article, show that the sinking-fund method at the rate i gives the formulas

$$D = \frac{C - S}{s_{\overline{n}|i}} s_{\overline{t}|i}, \qquad B = C - \frac{C - S}{s_{\overline{n}|i}} s_{\overline{t}|i}$$

EXERCISES

1. A sinking fund with annual deposits is created at ($j = .055$, $m = 1$) to replace in 5 years a machine which costs \$1850, and which has a scrap value of \$100. Construct the depreciation schedule.

2. Same as Exercise 1, except that ($j = .06$, $m = 2$).

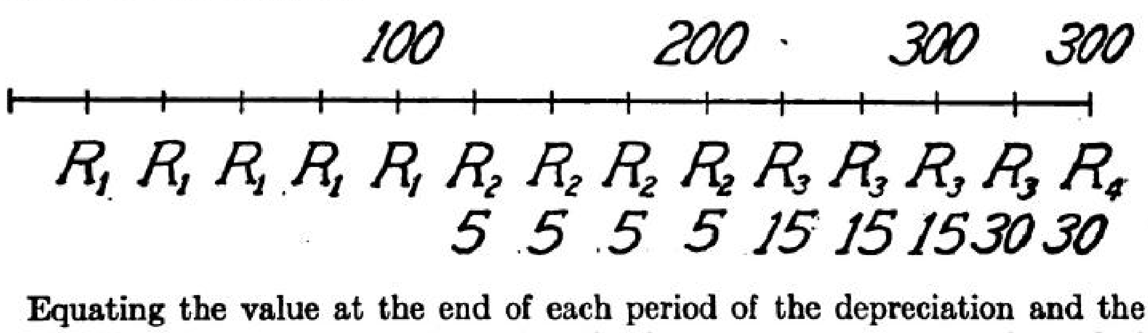
3. Same as Exercise 2, except that the deposits into the sinking fund are made semi-annually.

4. By the sinking-fund method, find the book value, at the end of the fifth year, of a building which costs \$100000 and which depreciates to \$70000 in 10 years. (Use $j = .05$, $m = 1$.) Ans. \$86820.61.

65. The appraisal method.* By the straight-line, compound-discount, and sinking-fund methods, an estimate is made of the depreciation charge of an article during its estimated term. In the *appraisal* method the depreciation term is divided into periods and an estimate is made of the depreciation charge for each period. The amount of depreciation for each period is often fairly well known from data at hand, and by properly utilizing this information it is possible to make a set of depreciation charges which are closer to the actual amounts of depreciation than it would be if no account whatever were taken of it. The straight-line, the constant percentage, or the sinking-fund method could be used to estimate the charges during any period; in this article, however, the sinking-fund method will be used. For each period after the first the interest payments on the fund already accumulated are used to help pay the depreciation charges for this period.

EXAMPLE. An article costing \$1000 has a scrap value of \$100 at the end of 14 years. It is estimated that this article will depreciate \$100 during the first five years, \$200 during the next four years, \$300 during the next three years, and \$300 during the last two years of the term. If the appraisal method is used with interest at 5% converted annually, find the depreciation charge at the end of each year and construct a depreciation schedule.

SOLUTION. In the following diagram, each section of which represents one year, the annual depreciation charge and the interest payments on the fund accumulated are shown below the line and the depreciation amounts for the periods into which the term is divided are shown above the line. In the diagram replace the last R_3 by R_4.



Equating the value at the end of each period of the depreciation and the interest payments during the period to the depreciation amount at the end of the period gives the following equations for determining R_1, R_2, R_3, and R_4.

* Communicated to the authors by Professor C. H. Forsythe.

$R_1 s_{\overline{5}|.05} = 100$; $(R_2+5)s_{\overline{4}|.05} = 200$; $(R_3+15)s_{\overline{3}|.05} = 300$; $(R_4+30)s_{\overline{2}|.05} = 300$

Solving, $R_1 = 18.10$; $R_2 = 41.40$; $R_3 = 80.16$; $R_4 = 116.34$

By use of these annual depreciation charges one can readily construct the following

DEPRECIATION SCHEDULE

YEAR	BOOK VALUE AT END OF YEAR	PAYMENT TO FUND AT END OF YEAR	INTEREST DUE ON FUND AT END OF YEAR	TOTAL IN FUND AT END OF YEAR
0	1000.00			
1	981.90	18.10		18.10
2	962.90	18.10	.90	37.10
3	942.95	18.10	1.85	57.05
4	922.00	18.10	2.85	78.00
5	900.00	18.10	3.90	*100.00*
6	853.60	41.40	5.00	146.40
7	804.88	41.40	7.32	195.12
8	753.72	41.40	9.76	246.28
9	700.01	41.40	12.31	*299.99*
10	604.85	80.16	15.00	395.15
11	504.93	80.16	19.76	495.07
12	400.02	80.16	24.75	*599.98*
13	253.68	116.34	30.00	746.32
14	100.03	116.34	37.31	*899.97*

EXERCISES

1. Solve the above example if the depreciation charges are made semi-annually and the sinking fund accumulates at $(j = .05, m = 2)$.

2. Solve the above example if the depreciation charges are made annually but the sinking fund accumulates at $(j = .05, m = 2)$. [In this case,

$$R_1 \frac{s_{\overline{10}|.05}}{s_{\overline{2}|.05}} = 100,\ R_2 \frac{s_{\overline{8}|.025}}{s_{\overline{2}|.025}} + 2.50\, s_{\overline{8}|.025} = 200, \text{ etc.]}$$

3. In the above example, show that the depreciation fund, D_3, and the book value, B_3, for the third period, are given by the formulas

$$D_3 = 300(1.05)^{t-9} + R_3 s_{\overline{t-9}|.05}$$
$$B_3 = 1000 - D_3$$

4. An article costing \$3000 has a value of \$400 at the end of eight years. It is estimated that this article will depreciate \$1000 during the first year, \$1000 during the next two years, and \$600 during the next five years. If the appraisal method is used with interest at 5% converted annually, find the depreciation charge at the end of each year and construct a depreciation schedule.

5. Solve Exercise 4 if the straight line method is used to estimate the annual depreciation charges for each period.

6. Solve Exercise 4 if the constant percentage method is used to estimate the annual depreciation charges for each period.

66. Graphs of book values and depreciation funds. Figure 9 shows the book values and the amounts in the depreciation fund for the examples solved in Arts. 62–65 inclusive.

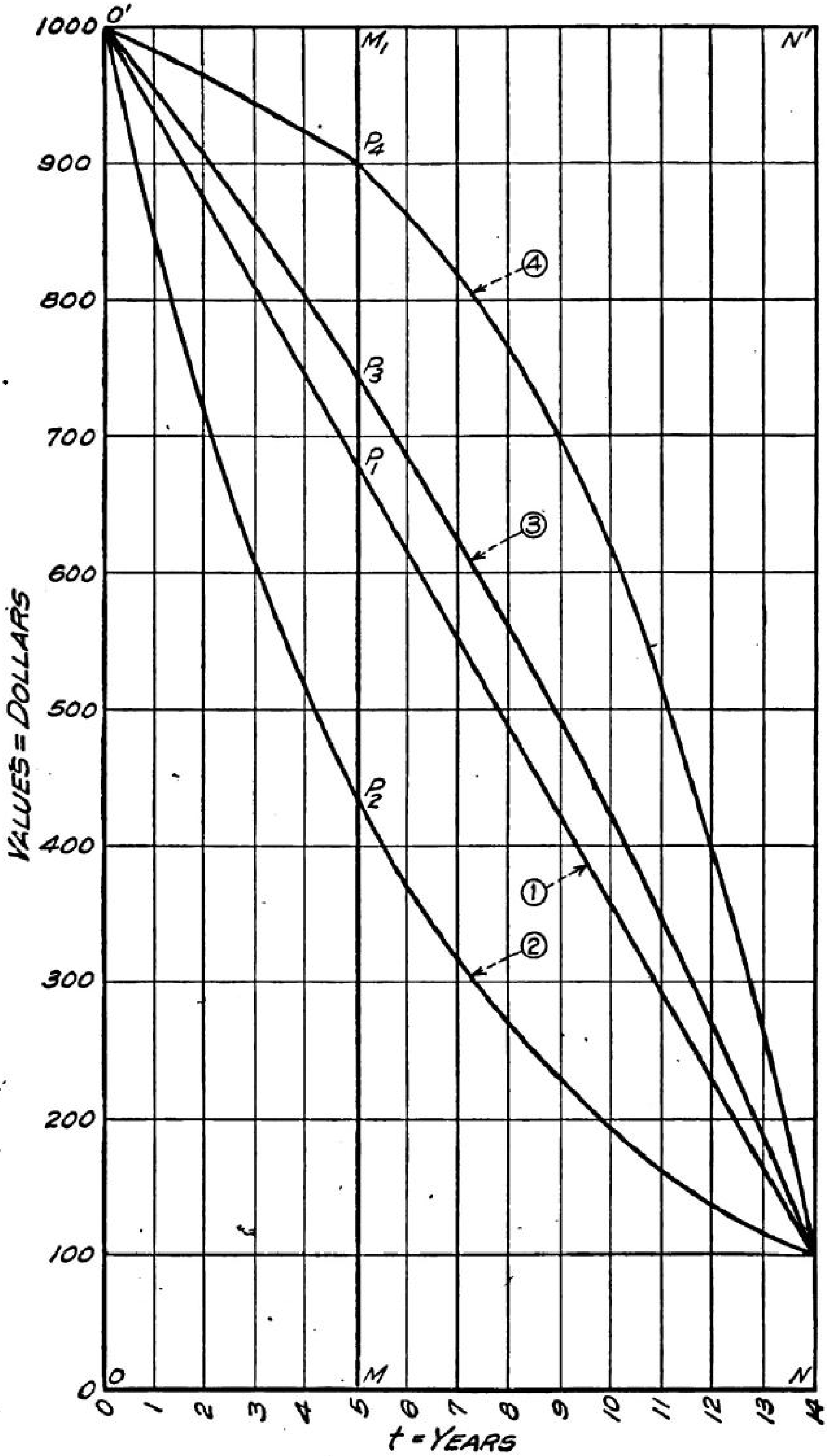
Graph (1) shows these values by the straight-line method, graph (2) by the constant percentage method, graph (3) by the sinking-fund method, and graph (4) by the appraisal method. For example, by the sinking fund method, MP_3 represents the book value of the article and M_1P_3 the amount in the depreciation fund at the end of five years. The slope at a point which corresponds to any time determines the rate at which the book value or the amount in the depreciation fund is changing at the time; a slope whose numerical value is large shows a rapid change while one whose numerical value is small shows a slow change.

EXERCISES

1. Construct the graph of book values of the article in Exercise 4, Art. 65.

2. Same as Exercise 1, for Exercise 5, Art. 65.

67. Other methods of estimating depreciation. *By the compound interest, or interest on investment method* charges are made at the end of each year to cover depreciation for the year and interest for the year on the book value of the article at the beginning of the year. The book values and the total depreciation charges are the same by this method as by the sinking-fund method. When the same interest rates are used on the book values and on



1000
900
800
700
600
500
400
300
200
100
0
O'
M1
N'
P4
P3
P1
P2
4
3
1
2
O
M
N
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
VALUES = DOLLARS
t = YEARS

FIGURE 9

the annual depreciation charges, the book value plus the depreciation charge is constant; when different rates are used, the book value plus the depreciation charge is not constant. Exercises 1 and 2 below illustrate depreciation schedules for the two cases.

By the *unit cost method* the value of a machine at any time is found by assuming that the net cost of a unit of its output equals that of a new machine which could replace it. The net cost of a unit for each machine is found by dividing the number of units it produces in a year into the sum of the annual charges for operating expenses, repairs, depreciation, and interest. Exercise 3 below illustrates the method by which the equation determining the value of the old machine is found.

It should be noted that each of the methods of estimating depreciation described in this and the preceding articles rests upon certain assumptions. In each method an assumption is made as to the amount of depreciation during the service life of an article or during an interval within the service life; the service life is also an assumed number of years. A further assumption is made as to the type of depreciation during the whole or a part of the service life. One type is given by the straight-line formula, one by the constant percentage, and another by the sinking fund. In practice that method should be used which in the judgment of the business accountant or evaluation engineer seems to be in best agreement with the data at hand and then the results determined should be modified to make them correspond to any additional data obtained later. The methods of mathematical statistics when applied to data which show with a fair amount of accuracy the depreciation which has taken place in an article for a term of years would lead to a more accurate determination of the depreciation charges of a like article than could ordinarily be obtained by the use of a method based upon assumptions which are in whole or in part *a priori*.

EXERCISES

1. An article costing \$1000 has a scrap value of \$100 at the end of 14 years. Complete the following compound-interest-method schedule in which ($j = .05$, $m = 1$), for book values and for annual depreciation charges:

DEPRECIATION SCHEDULE

Year	Book Value at End of Year	Depreciation Charges at End of Year	Total Depreciation at End of Year	Interest on Book Value at 5% at End of Year	Sum of Depreciation Charges at End of Year and Interest on Book Value
0	1000.00				
1	954.08	45.92	45.92	50.00	95.92
2	905.86	48.22	94.14	47.70	95.92
3	855.23	50.63	144.97	45.29	95.92
4					
5					
6					
7					
8					
9					
10					
11					
12					
13					
14					

2. An article costing $1000 has a scrap value of $100 at the end of 14 years. Complete the following compound-interest-method schedule in which the interest on book values is at (j = .08, m = 1) and the depreciation charge is at (j = .05, m = 1).

DEPRECIATION SCHEDULE

YEAR	BOOK VALUE AT END OF YEAR	DEPRECIATION CHARGES AT END OF YEAR AT 5%	TOTAL DEPRECIATION AT END OF YEAR	INTEREST ON BOOK VALUE AT 8%	SUM OF DEPRECIATION CHARGES AT END OF YEAR ON BOOK VALUES
0	1000.00				
1	954.08	45.92	45.92	80.00	125.92
2	905.80	48.22	94.14	76.33	124.55
3	852.70	50.63	147.30	72.46	123.09
4					
5					
6					
7					
8					
9					
10					
11					
12					
13					
14					

3. A machine whose cost of operation is \$500 per year and whose cost for annual repairs is \$200 has an estimated life of 5 years. A new machine whose cost of operation is \$450 per year and whose cost for annual repairs is \$100 may be purchased for \$1500 and has an estimated life of 8 years. If the scrap value of each machine is zero and if they produce the same number of units of annual output, find the value of the old machine at ($j = .06$, $m = 1$). If x is the value of the old machine, then

$$\frac{x}{s_{\overline{5}|.06}} + .06\,x + 500 + 200 = \frac{1500}{s_{\overline{8}|.06}} + (.06)1500 + 450 + 100$$

Solving, $$x = \$385.66$$

4. Same as Exercise 3 except that the annual output of the new machine is 10% greater than that of the old; 13% greater. Ans. \$82.54; \$2.06.

68. Composite life of a plant. The various parts of a plant do not ordinarily have the same depreciation term or lifetime. The composite life of such a plant may be defined as the number of years n for which the annual depreciation charge for all the parts equals the sum of the annual depreciation charge for the component parts. By means of this definition the equation determining the composite life of a plant can be readily formed.

EXAMPLE 1. Find the composite life under the sinking-fund method at 5% converted annually of a plant having the following parts:

PART	COST	SCRAP VALUE	LIFE
1	16000	400	10
2	23500	500	15
3	8600	200	8
4	12300	300	17

SOLUTION. The total wearing values of the parts are \$15600, \$23000, \$8400, and \$12000; the sum of these is \$59000. By definition the composite life, n years, by the sinking-fund plan, is determined by the equation

$$\frac{59000}{s_{\overline{n}|.05}} = \frac{15600}{s_{\overline{10}|.05}} + \frac{23000}{s_{\overline{15}|.05}} + \frac{8400}{s_{\overline{8}|.05}} + \frac{12000}{s_{\overline{17}|.05}}$$
$$= \$3650.19698$$
$$s_{\overline{n}|.05} = 16.1635113$$

Interpolation in Table V gives $n = 12.1$

EXERCISE. Solve the above equation for $\frac{1}{s_{\overline{n}|.05}}$ and find n by interpolation in Table VII.

EXAMPLE 2. Find the composite life of the plant in Example 1 by the straight-line method.

SOLUTION. In this case the equation which determines n is

$$\frac{59000}{n} = \frac{15600}{10} + \frac{23000}{15} + \frac{8400}{8} + \frac{12000}{17}$$
$$= \$4849.2157$$
$$n = 12.2$$

EXERCISE

Find the composite life of a manufacturing plant composed of the following units:

(*a*) building, costing \$100000 with a scrap value of \$10000 at the end of 20 years;

(*b*) heavy machinery, costing \$75000 with a scrap value of \$15000 at the end of 10 years;

(*c*) light machinery, costing \$25000 with a scrap value of \$1000 at the end of 5 years.

Use ($j = .06$, $m = 1$). Ans. 11.3.

69. Capitalized cost. The capitalized cost of an article is defined as the first cost plus the present value of perpetual renewals. When the renewals are made at periodic intervals and at equal cost they form a perpetuity. (Art. 29, Chapter II.)

EXAMPLE 1. A cab costing \$1600 must be replaced each 5 years. Find the capitalized cost if the scrap value of each cab is \$100. Use an interest rate of 5% converted annually.

SOLUTION. The renewals form a perpetuity whose rent is \$1500 and whose rent period is 5 years. The present value of this perpetuity is $\frac{1500}{.05}\frac{1}{s_{\overline{5}|.05}}$. Hence the capitalized cost, V, is given by

$$V = 1600 + \frac{1500}{.05}\frac{1}{s_{\overline{5}|.05}}$$
$$= \$7029.24$$

EXERCISE 1. Show that V, in Example 1, may be written in the form $V = 100 + \frac{1500}{.05}\frac{1}{a_{\overline{5}|.05}}$; compute its value from this form.

EXERCISE 2. If C is the first cost and D is the renewal cost of an article whose life is r years, show that the capitalized cost, V, at the rate i converted annually is given by

$$V = C - D + \frac{D}{i}\frac{1}{a_{\overline{r}|i}}$$

A definition of capitalized cost more general than the usual one given above may be stated as follows: The capitalized cost of an article is the first cost plus the present value of n' renewals. When n' becomes infinite, this definition reduces to the usual form. When the n' renewals are made at periodic intervals and at equal costs they form an annuity.

EXAMPLE 2. A cab costing \$1600 must be replaced every 5 years. If the scrap value is \$100, find the sum, V, of the first cost and the present value of 8 renewals. Use an interest rate of 5% converted annually.

SOLUTION. In this case

$$V = 1600 + 1500\frac{a_{\overline{40}|.05}}{s_{\overline{5}|.05}}$$
$$= \$6258.04$$

Two articles which will serve the same purpose may have equal or different capitalized costs. When they are different the one with the smaller cost should be used.

EXERCISES

1. Would it be more economical to use asbestos shingles which cost $12.00 per thousand and last 25 years or asphalt shingles which cost $7.00 per thousand and last 15 years? Assume that the cost of laying the shingles is the same in each case. Use $(j = .06, m = 1)$.

2. What is the capitalized cost of a bridge which must be replaced every 50 years at a cost of $350000? Use $(j = .055, m = 1)$. Ans. $375845.59.

3. What is the capitalized cost of an automobile which costs $1800, has a scrap value of $100, and a service life of 6 years. Use $(j = .05, m = 1)$.
Ans. $6798.59.

4. A trust fund for the perpetual maintenance of a hospital building is created. How much must be deposited with the trustees of the fund to provide $30000 for immediate use, $5000 at the end of each year for minor repairs, and $25000 at the end of each three years for major repairs. Use $(j = .05, m = 1)$. Ans. $288604.28.

5. A wood post which costs 45 cents and lasts 9 years can be set in soil for 10 cents. The post can be set in concrete for 25 cents, and it will then last 12 years. Find the capitalized cost of a post set in soil if 3 renewals are made, and that of a post set in concrete if 2 renewals are made. If a farmer wishes to maintain a fence for 36 years, which method of setting the posts is the more economical? Use $(j = .055, m = 1)$. Ans. $1.23; $1.26.

70. Capitalized cost equations. An equation which expresses equality in value between the capitalized costs of articles can often be used to find an unknown in capitalized cost problems. (See Art. 15, Chapter I.) In this article two types of capitalized cost equations are presented. In one type two articles have equal capitalized costs; in the other the capitalized cost of one article equals h times that of the other.

EXAMPLE 1. A transfer company is using a truck having $3000 for its cost, 3 years for its service life, and zero for its scrap value. What can it afford to pay for another truck having 5 years for its service life and zero for its scrap value. Use an interest rate of 6% converted annually.

SOLUTION. Let x denote the amount that can be paid for another truck to make its capitalized cost the same as that of the one the company is using. Equating the capitalized costs gives [Exercise 2, Example 1, Art. 69]

$$\frac{x}{.06}\frac{1}{a_{\overline{5}|.06}} = \frac{3000}{.06}\frac{1}{a_{\overline{3}|.06}}$$

Solving,
$$x = 3000\, a_{\overline{5}|.06} \cdot \frac{1}{a_{\overline{3}|.06}}$$
$$= \$5727.66$$

It follows that the company can pay an amount not exceeding \$5727.66 for another truck.

EXERCISE 1. An article having $C + x$ for its first and its renewal cost and $r + k$ for its service life will serve the same purpose as an article having C for its first and its renewal cost and r for its service life. By solving the capitalized cost equation $\frac{C+x}{i}\frac{1}{a_{\overline{r+k}|i}} = \frac{C}{i}\frac{1}{a_{\overline{r}|i}}$ for x, show that

$$x = C\,\frac{a_{\overline{r+k}|i} - a_{\overline{r}|i}}{a_{\overline{r}|i}}$$
$$= C\,\frac{a_{\overline{k}|i}\,(1+i)^{-r}}{a_{\overline{r}|i}} \qquad (k > 0)$$
$$= C\,\frac{a_{\overline{k}|i}}{s_{\overline{r}|i}}$$

This type of equation is useful in finding the amount that one is justified in expending in order to extend the life of an article.

EXAMPLE 2. A machine costing \$1000 has a service life of 10 years and a scrap value of \$100. How much would one be justified in expending on the machine to double its output and decrease its service life by 2 years, the scrap value remaining the same. Use an interest rate of 5% converted annually.

SOLUTION. Let x denote the amount that can be expended on the machine to make its capitalized cost twice that of the original machine. Then

$$100 + \frac{900 + x}{.05}\frac{1}{a_{\overline{8}|.05}} = 2\left(100 + \frac{900}{.05}\frac{1}{a_{\overline{10}|.05}}\right)$$

Solving,
$$x = \$638.94$$

It follows that one can afford to expend an amount not exceeding \$638.94.

EXERCISE 2. An article having $C + x$ for its first and its renewal cost, and $r + k$ for its service life will serve the same purpose as h articles each of which has C for its first and its renewal cost and r for its service life. By solving the capitalized cost equation $\frac{C+x}{i}\frac{1}{a_{\overline{r+k}|i}} = h\left(\frac{C}{i}\frac{1}{a_{\overline{r}|i}}\right)$ for x, show that

$$x = C(h-1) + h\,C \cdot \frac{a_{\overline{r+k}|i} - a_{\overline{r}|i}}{a_{\overline{r}|i}}$$
$$= C(h-1) + hC \cdot \frac{a_{\overline{k}|i}}{s_{\overline{r}|i}} \qquad (k > 0)$$
$$= C(h-1) - hC \cdot \frac{s_{\overline{-k}|i}}{s_{\overline{r}|i}} \qquad (k < 0)$$

This type of equation is useful in finding the amount one is justified in spending to change the service life and multiply the productivity of an article.

EXERCISES

1. Wood posts which cost 45 cents and last 9 years will, if treated with creosote, last 15 years. How much could a farmer afford to spend treating each post with creosote? Use (j = .06, m = 1). Ans. .19.

2. A machine costing $1000 has a service life of 10 years and a scrap value of $100. How much would one be justified in expending on the machine to increase its life to 12 years? Use (j = .05, m = 1). Ans. $133.05.

3. Same as Exercise 2 except that in addition to increasing the life of the machine, its productivity is increased by 20%. Ans. $348.52.

4. An automobile has a value of $300, a life of 2 years, and a scrap value of $25. It is estimated that replacements which would cost $100 would increase its life to 4 years. Would it be economical to have the replacements made? Use (j = .055, m = 1).

5. A wood post which costs 45 cents and lasts 9 years can be set in soil for 10 cents. If the post set in concrete will last 12 years, how much can a farmer afford to spend setting the post in concrete if he wishes to maintain a fence for 36 years? Use (j = .055, m = 1).

MISCELLANEOUS EXERCISES

1. On March 3, A gave B two demand notes, one for $127.50, the other for $1325.60. Each note bore ordinary simple interest from date, the first at 6½%, the second at 6%. On April 1, B offered A $25.00 if he would refund the notes on that day. A accepted the offer and gave to a bank a note for 90 days. If the bank charged 7% ordinary simple discount and A used the $25.00 to reduce the amount necessary to refund the debt, find the face of the note given to the bank. On the assumption that A could not have paid the two notes before June 30 how much did he gain or lose by the transaction?

2. A bank paid $676.85 for a note for $735.00 due in one year. What rate of interest did the investment yield the bank? Find the corresponding rate of discount.

3. Given the equation $\dfrac{1}{1-d} = 1 + i$. Plot the graph after completing the table of values given below:

i	0	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10	1	2	-1	-2	∞
d	0															

Is there a value of d corresponding to every arbitrarily chosen positive value for i? What portion of the graph is of importance in the solution of problems in interest and discount? At what point does the line $i = d$ cut the hyperbola? Interpret $i = d = 0$.

4. A man pays \$7800 for a house and rents it at \$65.00 per month payable in advance. Taxes and upkeep cost \$273.00 annually and are charged against the investment at the end of the year. What ordinary simple interest rate does the owner make if he is able to invest each monthly rental payment as made at 5% ordinary simple interest?

5. A merchant buys a bill of goods for \$1275.00. He has the option of paying cash with a 2% discount, or net in 60 days. What ordinary simple discount rate could he afford to pay a bank in order to make cash payment, and how much would it be necessary for him to borrow if he did so?

6. Same as Exercise 5 except that $1\frac{1}{2}$% discount is allowed for cash in 10 days.

7. The list price of a bill of goods is \$565.85 with 10% and 5% off. If an additional 2% is given for cash, would the merchant be justified in borrowing from a bank at 8% ordinary simple discount to pay the bills rather than to pay net at end of 60 days. State the difference between the two methods of payment at the end of the 60 days.

8. The price of a bill of goods is \$125.00 cash or \$130.00 in 60 days. On the basis of 6% ordinary simple discount, find how much better from the purchaser's standpoint is the cash price on the date of sale; how much better at the end of the 60 days.

9. Two equal sums of money were placed at the same time on interest, the first at ($j = .04$, $m = 4$), the second at ($j = .05$, $m = 2$). When will the second sum amount to $\frac{3}{2}$ of the first?

10. A owes B \$1800.00 due in 3 years with interest at 5% payable semi-annually. Two years before it is due B sells the note to C at such a price that C makes ($j = .065$, $m = 2$) on his investment. How much did C pay for the note?

11. A house is offered for \$2000 cash, \$1000 at the end of the first year, \$800 at the end of the second, and \$500 at the end of the third year, or \$1000 cash, \$1500 at the end of the first year, \$1200 at the end of the second, and \$1000 at the end of the third. What is the difference between the two options on the day of sale if an interest basis of ($j = .06$, $m = 2$) is used?

12. A debt of \$3000 was extinguished by three annual payments, \$800 at the end of the first year, \$1000 at the end of the second, and x at the end of the third. Find x if the interest rate for the first two years was ($j = .06$, $m = 1$) and ($j = .055$, $m = 1$) for the third year.

13. By the will of his father, a boy aged 12 was left an annuity of \$25 payable at the end of each month until he becomes of age, and an annuity of \$500 payable at the end of each year thereafter until he becomes 30 years of age. On the assumption that the boy will certainly live until he is 30 years old, find the value of future payments on his twelfth, his eighteenth, and his twenty-fifth birthdays. Use ($j = .05$, $m = 2$).

14. The annual premium on a certain non-participating life insurance policy for $1000 was $45.10 payable at the beginning of each year. Immediately after the nineteenth premium was paid, the holder of the policy died. If, instead of taking out the policy, the money had been deposited in a savings bank which pays 4% converted quarterly, how much more would the estate have received from the bank than it did receive from the insurance company? If the policyholder had died immediately after the payment of the fifth premium, how much more would the estate have received from the insurance company than it would have received from the savings bank? When would the deposits in the savings bank have had the same value as the face value of the policy?

15. In order to provide for the education of his daughter, a father wishes to provide an annuity of $1500 at the beginning of each of her four years at college. How much must be deposited each month in a savings bank which pays ($j = .04$, $m = 4$) if the first deposit is made at the time of the child's birth, and the last on her eighteenth birthday when the first $1500 is to be available?

16. Semi-annual instalments of $270.00, including interest and principal, must be made on a note for $4500 which bears interest at 6% payable semi-annually. Find the number of instalments and the amount of the last one.

17. Had the note in Exercise 16 been sold for $4100 immediately after the second instalment had been paid, what rate of interest would the purchaser of the note have made if $m = 2$?

18. Find the selling price of the note in Exercise 16 immediately after the payment of the second instalment to yield ($j = .065$, $m = 2$).

19. A man wishes to borrow $5000 for 5 years. He has the option of borrowing it from an insurance company at 5½% payable semi-annually or from a building and loan company at 6% payable semi-annually. If he chooses to borrow from the insurance company, he must pay a 2% commission to the agent who makes the loan. The loan company does not charge a commission. How much would he save by borrowing from the insurance company if he can use money at ($j = .07$, $m = 2$) in his business?

20. A company purchased 2250 acres of coal land at $150 per acre. For the ten years before development took place the tax was $3000 at the end of each year. It is estimated that (1) the life of the mine will be 45 years, (2) the recovery per acre will be 6000 tons, (3) the plant, including buildings and equipment, which cost $400,000, must be replaced every 15 years, (4) the shafts and tunnels, which cost $125,000, will last during the life of the mine, and (5) the value of the land after the coal is exhausted will be $50 per acre. If the annual charge for wages is $357,000, that for administration, repairs, and taxes is $153,000, and the average selling price per ton of the coal at the mine is $2.15, find how much the company had invested in the mine 5 years after development took place, and its value at that time. Use ($j = .05$, $m = 1$) and

proceed on the hypthesis that the expenditures for each year are made at the end of the year, that the yearly output of coal is sold at that time, and that the annual production is constant during the life of the mine.

21. What profit would the coal company of Exercise 20 have made had the mine been leased immediately after the development was completed on a basis of 25 cents a ton royalty and an annual production of 300,000 tons for 45 years? What would have been the value at that time of the future profits of the lessee? Use (j = .05, m = 1).

22. Find the amount the coal company in Exercise 20 had invested in the mine at the end of the tenth year of its development. What was the value of the mine at that time if the price of coal at the mine advanced to $2.20 per ton for the remaining 35 years of the life of the mine? Use (j = .05, m = 1).

23. It is estimated that a slope opening of a certain mine, which would cost $500,000, could produce coal at $1.90 per ton; that a shaft opening, which would cost $600,000, could produce coal at $1.86 per ton. It is also estimated that the mine will produce 200,000 tons of coal per year for 40 years. Which method of developing the mine is the better if an interest rate of (j = .05, m = 1) is used?

24. The cost of a mine tie is 20 cents and its service life 3 years. If the tie is dipped in creosote, its life is 12 years. How much can a mine company afford to spend treating each tie with creosote? Use (j = .05, m = 1) and assume that the mine will be operated for 36 years.

CHAPTER IV

LIFE ANNUITIES AND LIFE INSURANCES

71. Probability. If two coins be tossed they can fall in any one of four ways: both heads; both tails; the first, head, the second, tail; the first, tail, the second, head. It will be assumed that these ways are equally likely. The event "one head and one tail" can happen in two ways. The probability that the event "one head and one tail" will happen is $\frac{2}{4}$, the numerator being the number of ways in which the event can happen and the denominator being the number of ways in which it can happen plus the number in which it can fail. In like manner the probability that both will be heads is $\frac{1}{4}$. These simple examples illustrate the following *definition of probability:*

If an event can happen in h ways and fail in f ways, and if the $h + f$ ways are equally likely, the probability that the event will happen is $\frac{h}{h+f}$ and the probability that it will fail is $\frac{f}{h+f}$.

Since $\frac{h}{h+f} + \frac{f}{h+f} = 1$, it follows that the probability that an event will happen plus the probability that it will fail is unity.

If two coins be tossed, each can fall in two ways. Since to each of the ways in which the first can fall there correspond two ways in which the second can fall, it follows that the two can fall in $2 \cdot 2 = 4$ ways. If a coin and a die be tossed, they can fall in $2 \cdot 6 = 12$ ways, since to each of the two ways in which the coin can fall there correspond six ways in which the die can fall. If two coins and a die be tossed, they can fall in $2 \cdot 2 \cdot 6 = 24$ ways, since to each of the $2 \cdot 2 = 4$ ways in which the coins can fall there correspond six ways in which the die can fall. Similar reasoning leads to the

Fundamental Principle. *If one thing can be done in m_1 ways, and if, after it is done, a second thing can be done in m_2 ways, the two*

things taken together can be done in the order stated in m_1m_2 ways; more generally, if one thing can be done in m_1 ways, a second in m_2 ways, a third in m_3 ways, and so on, the number of ways in which they can be done when taken all together in the order stated is $m_1m_2m_3 \cdots$.

By the number of permutations or arrangements of n things taken r at a time is meant the number of arrangements consisting of r things which can be formed from n different things. For example, twelve permutations of two letters can be formed from the four letters a, b, c, d; these twelve permutations are *ab*, *ba*, *ac*, *ca*, *ad*, *da*, *bc*, *cb*, *bd*, *db*, *cd*, and *dc*. That there are twelve such arrangements follows at once from the above fundamental principle, since the first letter can be chosen in four ways and after it is selected the second can be chosen in three ways. If ${}_nP_r$ denotes the number of permutations of n different things taken r at a time it follows from the same principle that

$${}_nP_r = n(n-1)(n-2)\cdots(n-r+1).$$

By the number of combinations of n things taken r at a time is meant the number of different sets of r things which can be formed from n different things. For example, six combinations of two letters can be formed from the four letters, a, b, c, d; these six combinations are *ab*, *ac*, *ad*, *bc*, *bd*, and *cd*. That there are six such combinations may be seen as follows: To each combination there correspond two and just two permutations; for example, to the combination *ab* correspond the permutations *ab* and *ba*. Since there are twelve permutations of four letters taken two at a time, there must be $\frac{12}{2} = 6$ combinations of these letters taken two at a time. In general to a combination of r different things there correspond ${}_rP_r = r(r-1)\cdots 2\cdot 1 = r!$ permutations of them. If ${}_nC_r$ denotes the number of combinations of n different things taken r at a time, it follows that ${}_nC_r \cdot r! = {}_nP_r$ or

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 2\cdot 1} = \frac{n!}{r!(n-r)!}$$

The fundamental principle and the formula for ${}_nC_r$ are often useful in determining a probability. Other useful results are presented in Art. 73.

Example. From a bag containing 6 white and 4 red balls 5 are drawn at random. Find the probability that 3 are white and 2 are red.

Solution. By the formula for ${}_nC_r$, 5 balls can be selected from 10 in $\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 252$ ways; 3 white balls can be selected from 6 white in $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 20$ ways; and 2 red balls can be selected from 4 red in $\frac{4 \cdot 3}{1 \cdot 2} = 6$ ways. Hence, by the fundamental principle, 3 white and 2 red balls can be selected from 6 white and 4 red in $20 \cdot 6 = 120$ ways. It follows that the probability that 3 are white and 2 are red is $\frac{120}{252} = \frac{10}{21}$.

EXERCISES

1. There are 4 paths up a mountain. In how many ways can one ascend the mountain and then descend by a different path? Ans. 12.

2. If there are 4 railroad lines from Columbus to Chicago and 5 from Chicago to St. Louis, in how many ways can one go from Columbus to St. Louis, through Chicago? Ans. 20.

3. How many committees can be appointed from 5 men and 4 women if each committee is made up of 2 men and 2 women? Ans. 60.

4. How many numbers, each of two digits, can be made if the first digit is chosen from the digits 1, 2, 3, and 4 and the second from 5, 6, 7, 8, and 9? Ans. 20.

5. How many numbers, each of four digits, can be made if the first two are chosen from 1, 2, 3, and 4, and the last two from 5, 6, 7, 8, and 9, and if there is no repetition of the digits? How many numbers can be made if there is no restriction as to the repetition of the digits? Ans. 240; 400.

6. If a bag contains 7 white and 5 red balls, and 3 balls are drawn at random, what is the probability that 2 are white and 1 is red? Ans. $\frac{105}{220}$.

7. If two dice are thrown, what is the probability of two threes; two fours; a three paired with a four? Ans. $\frac{1}{36}$; $\frac{1}{36}$; $\frac{1}{18}$.

8. If 3 coins are thrown, what is the probability of 3 heads; 2 heads and 1 tail; 1 head and 2 tails; 3 tails? Ans. $\frac{1}{8}$; $\frac{3}{8}$; $\frac{3}{8}$; $\frac{1}{8}$.

9. If a bag contains 3 red and 4 white balls, and another bag 4 red and 3 white balls, what is the probability of getting one red and one white ball in two drawings, one from each bag? Ans. $\frac{25}{49}$.

72. Probability derived from observation. In the example and exercises in Art. 71 the probabilities are derived in each case by an *a priori* determination of all the equally likely ways in which an event can happen or fail. This method is well suited to find probabilities pertaining to games of chance, in which field many

contributions to the theory of probability have had their origin. In life insurance, life annuities, mathematics of statistics, and other fields in which the notion of probability is important, it is impossible to make an *a priori* determination of the equally likely ways. In such cases the probability of an event is determined empirically by observing in what proportion of cases the event happens on a large number of occasions.

If h denotes the number of times an event happens on n occasions chosen at random (n, large), then, in the absence of further information, $p = \frac{h}{n}$ is taken as the best estimate of the probability that the event will happen on a given occasion.

It follows from this definition that *if p denotes the probability that an event will happen, $1 - p$ denotes the probability that it will fail.* For, out of n occasions, the event will happen $pn (= h)$ times and will fail $n - pn = (1 - p)n$ times. Hence, by the definition, the probability that the event will fail on a given occasion is $\frac{(1 - p)n}{n} = 1 - p.$

In the table * on page 158 are given, for the years 1890 to 1916 inclusive, the number of revenue passenger miles for railroads of all classes, the number of revenue passengers killed and injured, together with the probability of death and injury per thousand passenger miles.

EXERCISES

1. Verify, for the year ending June 30, 1901, the probability of death and injury per thousand passenger miles as given in the above table.

2. Find, for the entire period covered by the above table, the probability of death and injury per thousand passenger miles.

73. Independent, dependent, and mutually exclusive events. Two or more events are said to be dependent or independent according as the occurrence of any one of them does or does not affect the occurrence of the others. Two or more events are said to be mutually exclusive when the occurrence of any one of them

* Accident Bulletin, Number 92, Interstate Commerce Commission.

YEAR ENDING	REVENUE PASSENGER MILES, ROADS OF ALL CLASSES	PASSENGERS KILLED	PROBABILITY OF DEATH PER 1000 PASSENGER MILES	PASSENGERS INJURED	PROBABILITY OF INJURY PER 1000 PASSENGER MILES
June 30, 1890	11,847,785,617	286	.000024	2425	.00020
June 30, 1891	12,844,243,881	293	.000023	2972	.00023
June 30, 1892	13,362,898,299	376	.000028	3227	.00024
June 30, 1893	14,229,101,084	299	.000021	3229	.00023
June 30, 1894	14,289,445,893	324	.000023	3034	.00021
June 30, 1895	12,188,446,271	170	.000014	2375	.00019
June 30, 1896	13,049,007,233	181	.000014	2873	.00026
June 30, 1897	12,256,939,647	222	.000018	2795	.00023
June 30, 1898	13,379,930,004	221	.000017	2945	.00023
June 30, 1899	14,591,327,613	239	.000016	3442	.00024
June 30, 1900	16,038,076,200	249	.000016	4128	.00026
June 30, 1901	17,353,588,444	282	.000016	4988	.00029
June 30, 1902	19,689,937,620	345	.000012	6683	.00034
June 30, 1903	20,915,763,881	355	.000017	8231	.00039
June 30, 1904	21,923,213,536	441	.000020	9111	.00042
June 30, 1905	23,800,149,436	537	.000023	10457	.00044
June 30, 1906	25,167,240,831	359	.000014	10764	.00043
June 30, 1907	27,718,554,030	610	.000022	13041	.00047
June 30, 1908	29,082,836,944	381	.000013	11556	.00040
June 30, 1909	29,109,322,589	253	.000009	10311	.00035
June 30, 1910	32,338,496,329	324	.000010	12451	.00039
June 30, 1911	33,201,694,699	299	.000009	12042	.00036
June 30, 1912	33,132,354,783	283	.000009	14938	.00045
June 30, 1913	34,672,685,424	350	.000010	15130	.00044
June 30, 1914	35,357,221,302	232	.000007	13887	.00039
June 30, 1915	32,474,923,456	199	.000006	10914	.00034
June 30, 1916	35,220,015,651	239	.000007	7488	.00021

excludes the occurrence of any other. In this article three elementary theorems are presented which are useful in determining probabilities of events of these types. Many other theorems are given in works on probability.

Independent events. *If $p_1, p_2, \cdots p_r$ are the separate probabilities of r independent events, the probability that they all happen on a given occasion when all of them are in question is $p_1 p_2 \cdots p_r$.*

By the definition in Art. 72, out of n occasions in which all the events are in question, the first event will happen $p_1 n$ times; out of these $p_1 n$ occasions the second event will happen $p_2(p_1 n) = p_1 p_2 n$ times. That is, out of n occasions the first two events will happen $p_1 p_2 n$ times. Continuing this process it is seen that out of n occasions all the r events will happen $p_1 p_2 \cdots p_r n$ times. Hence, by Art. 72, $p_1 p_2 \cdots p_r$ is the probability that all the events will happen on a given occasion.

Exercise. Use the definition and the fundamental principle in Art. 71 to prove this theorem.

Dependent events. *If p_1 is the probability of a first of r events, p_2 is the probability of a second event after the first has happened, p_3 is the probability of a third event after the first and second have happened, and so on, then $p_1 p_2 \cdots p_r$ is the probability that the r events will happen in the order specified.*

The proof of this theorem is entirely analogous to that for independent events.

Mutually exclusive events. If $p_1, p_2, \cdots p_r$ *are the separate probabilities of r mutually exclusive events, the probability that one of these events will happen on a given occasion when all of them are in question is $p_1 + p_2 \cdots + p_r$.*

By the definition in Art. 72, out of n occasions in which all of the events are in question, the r events will happen $p_1 n, p_2 n, \cdots p_r n$ times respectively. Since only one of these events can happen on a given occasion, it follows that out of n occasions one or the other of r events will happen $(p_1 n + p_2 n + \cdots + p_r n) = (p_1 + p_2 + \cdots p_r)n$ times. Hence $p_1 + p_2 + \cdots + p_r$ is the probability that one of the events will happen on a given occasion.

An illustration of the first and last of the above theorems is afforded by two lives, A and B, of ages x and y respectively. If p_x denotes the probability that A lives one year and p_y the probability that B lives one year, then $1 - p_x$ is the probability that A dies within one year and $1 - p_y$ is the probability that B dies within one year. By the first theorem the probabilities of the pairs of independent events, A lives B lives, A lives B dies, A dies B lives, A dies B dies are $p_x p_y$, $p_x(1 - p_y)$, $(1 - p_x)p_y$, $(1 - p_x)(1 - p_y)$ respectively. By the last theorem the probability that one or the other of the mutually exclusive events, A lives B lives, A dies B dies, happens, is $p_x p_y + (1 - p_x)(1 - p_y)$, and the probability that some one of the four mutually exclusive pairs of events happens is $p_x p_y + p_x(1 - p_y) + (1 - p_x)p_y + (1 - p_x)(1 - p_y)$. This last sum is unity, which it should be, since some one of the four pairs must happen.

EXERCISES

1. A traveler has two railroad connections to make. The probability of making the first is $\frac{2}{3}$, of the second $\frac{2}{5}$. What is the probability of his making both connections? Ans. $\frac{4}{15}$.

2. If a bag contains 8 white and 4 black balls, and one ball at a time is drawn from the bag, what is the probability of drawing 2 white balls in 2 drawings; 2 black balls in 2 drawings; 1 white and 1 black in 2 drawings?
Ans. $\frac{14}{33}$; $\frac{1}{11}$; $\frac{16}{33}$.

3. If a bag contains 10 white and 8 black balls, and one ball at a time is drawn from the bag, what is the probability of drawing 2 white balls in 2 drawings; 3 black in 4 drawings; 2 white followed by 2 black in 4 drawings; 1 white followed by 3 black in 4 drawings; 3 black followed by 1 white in 4 drawings; 3 white in 4 drawings? Ans. $\frac{5}{17}$; $\frac{28}{153}$; $\frac{7}{102}$; $\frac{7}{153}$; $\frac{7}{153}$; $\frac{16}{51}$.

4. A purse contains 9 dimes and a nickel; a second purse contains 10 dimes. Nine coins are chosen at random from the first purse and placed in the second, and then nine coins are chosen at random from the second and placed in the first. Show that the probability that the nickel is in the first purse is $\frac{10}{19}$.

5. If two dice are thrown, show that the probability of a seven[1] is $\frac{1}{6}$; of a six is $\frac{5}{36}$; of a two is $\frac{1}{36}$; show that the probability of a three is the same as that for an eleven.

6. If two dice are thrown, show that the probability of a five or less is $\frac{5}{18}$.

[1] A seven is thrown when the sum of the numbers on the top of the dice is seven.

74. Mortality Tables. A mortality table is a table which shows for a large group of persons of the same age the numbers living at consecutive ages and the number dying between each pair of consecutive ages. The interval between consecutive ages is ordinarily one year except for infant mortality in which case it is one month. The number of persons assumed to be living at the first age given in a mortality table is called the *radix* of the table. Many mortality tables have been constructed from data obtained principally from population and insurance statistics. Glover's United States Tables, published in 1910, are based on population statistics. The American Experience Table (Table XI), published in 1868, is based on insurance statistics. This table is widely used in the United States and it will be used in what follows unless otherwise stated. In this table the radix is 100,000.

A standard notation is used with mortality tables. In this notation,

l_x denotes the number living at age x,

d_x denotes the number dying between ages x and $x + 1$,

p_x denotes the probability that a person aged x will live one year,

${}_np_x$ denotes the probability that a person aged x will live n years,

q_x denotes the probability that a person aged x will die within one year,

$|{}_nq_x$ denotes the probability that a person aged x will die within n years,

${}_n|q_x$ denotes the probability that a person aged x will die between the ages $x + n$ and $x + n + 1$.

The values of l_x, d_x, p_x, and q_x are given in Table XI. The expressions for the values of the above symbols in terms of the values of l_x can be written at once from their definitions and from the definition of probability given in Art. 72. For example,

$$d_{20} = l_{20} - l_{21}, \quad p_{20} = \frac{l_{21}}{l_{20}}, \quad q_{20} = \frac{l_{20} - l_{21}}{l_{20}}, \quad {}_{|10}q_{20} = \frac{l_{20} - l_{30}}{l_{20}},$$

$$_{10|}q_{20} = \frac{l_{30} - l_{31}}{l_{20}}.$$

By means of the theorems in Art. 73 various other probabilities can be readily expressed in terms of the above symbols. For example, the probability that two persons aged x and y will both survive $n - 1$ years but will not both survive n years is $_{n-1}p_x \cdot {}_{n-1}p_y - {}_np_x \cdot {}_np_y$. This may be seen as follows: By the theorem on independent events, Art. 73, $\frac{l_{x+n-1}}{l_x} \cdot \frac{l_{y+n-1}}{l_y}$ is the probability that both will survive $n - 1$ years, or that they will attain the ages $x + n - 1$, $y + n - 1$. Having attained these ages, by the same theorem, the probability that both will live another year is $\frac{l_{x+n}}{l_{x+n-1}} \cdot \frac{l_{y+n}}{l_{y+n-1}}$. So that $1 - \frac{l_{x+n}}{l_{x+n-1}} \cdot \frac{l_{y+n}}{l_{y+n-1}}$ is the probability that both will not live another year. By the theorem on dependent events, Art. 73, it now follows that the desired probability is

$$\frac{l_{x+n-1}}{l_x} \cdot \frac{l_{y+n-1}}{l_y}\left(1 - \frac{l_{x+n}}{l_{x+n-1}} \cdot \frac{l_{y+n}}{l_{y+n-1}}\right) = \frac{l_{x+n-1}}{l_x} \cdot \frac{l_{y+n-1}}{l_y} - \frac{l_{x+n}}{l_x} \cdot \frac{l_{y+n}}{l_y}$$
$$= {}_{n-1}p_x \cdot {}_{n-1}p_y - {}_np_x \cdot {}_np_y.$$

For other methods of proof see Exercise 11 below.

EXERCISES

1. Compute the values of p_{25} and of q_{25}. Use Table XI in this exercise and in those Exercises following.

2. Compute the probability that a person aged 25 will live 14 years; 15 years. Ans. .885771; .877280.

3. Compute the probability that a person aged 25 will die within 14 years; within 15 years. Ans. .114229; .122720.

4. Compute the probability that a person aged 25 will die in the 15th year. Ans. .008491.

5. Same as Exercises 2, 3, and 4 for a person aged 30.
Ans. .877623; .868120; .122377; .131880; .009504.

6. Compute the probability that two persons aged 25 and 30 will live (*a*) 14 years; (*b*) 15 years. Ans. .777374; .761584.

7. Compute the probability that two persons aged 39 and 44 will not both live one year. Ans. .020311.

8. Compute the probability that a person aged 25 will live 15 years and a person aged 30 will die in the 15th year. Ans. .008337.

9. Compute the probability that a person aged 30 will live 15 years and a person aged 25 will die in the 15th year. Ans. .007371.

10. Compute the probability that two persons aged 25 and 30 will both die in the 15th year. Ans. .000081.

11. Compute the probability that two persons aged 25 and 30 will both live 14 years and that at least one will die in the 15th year, by the methods:

(*a*) Divide the number of pairs living at ages 25 and 30, $(l_{25} \cdot l_{30})$, into the difference between the number of pairs living 14 and 15 years hence, $(l_{39} \cdot l_{44} - l_{40} \cdot l_{45})$,

(*b*) Find the product of the results found in Exercises 6 (*a*) and 7,

$$\frac{l_{39}}{l_{25}} \cdot \frac{l_{44}}{l_{30}} \left(1 - \frac{l_{40}}{l_{39}} \cdot \frac{l_{45}}{l_{44}}\right),$$

(*c*) Subtract the result found in Exercise 6 (*b*) from that found in 6 (*a*),

$$\frac{l_{39}}{l_{25}} \cdot \frac{l_{44}}{l_{30}} - \frac{l_{40}}{l_{25}} \cdot \frac{l_{45}}{l_{30}},$$

(*d*) Find the sum of the results found in Exercises 8, 9, and 10,

$$\frac{l_{40}}{l_{25}} \cdot \frac{l_{44} - l_{45}}{l_{30}} + \frac{l_{45}}{l_{30}} \cdot \frac{l_{39} - l_{40}}{l_{25}} + \frac{l_{44} - l_{45}}{l_{30}} \cdot \frac{l_{39} - l_{40}}{l_{25}}.$$

Ans. .015790.

12. Draw the graph of the curve showing the probability of dying for each year listed in Table XI.

13. Draw the graph of the curve showing the number of deaths for each year listed in Table XI. In what year do the greatest number of deaths take place?

75. The expectation of life. By the expectation of life at age x is meant the average number of years to be lived by persons of age x. If the deaths during any year are assumed to take place at the beginning of the year, then l_x persons of age x will live l_{x+1} years during the first year (since l_{x+1} of the l_x persons live throughout this year), l_{x+2} years during the second, and so on to the end of the table. Under this assumption the expectation of life at age x, called the *curtate expectation*, e_x, is given by

$$e_x = \frac{l_{x+1} + l_{x+2} + \cdots}{l_x}$$

Similarly, under the assumption that the deaths during any year take place at the end of the year, the expectation of life at age x is given by

$$\frac{l_x + l_{x+1} + \cdots}{l_x}.$$

Under the assumption that the deaths during any year are distributed uniformly throughout the year, so that on the average each person will live half a year in the year of death the expectation of life at age x, called the *complete expectation* and denoted by $\mathring{e}_x$, is given by adding $\frac{1}{2}$ to the curtate expectation. It follows that

$$\mathring{e}_x = \frac{1}{2} + \frac{l_{x+1} + l_{x+2} + \cdots}{l_x}.$$

It is often supposed by those unacquainted with actuarial methods, that the expectation of life is used as a basis for actuarial computations. That this is not the case will be seen in the subsequent articles. The expectation of life is sometimes used to determine the approximate value of a life interest in settling estates.

EXERCISE

Compute the complete expectation of life at the ages 35, 43, 67, and 71.
Ans. 31.78; 26.00; 10.00; 8.00.

76. The value of an expectation. By the value at a given time, of the expectation of receiving a sum of money, S, due in n years, $(n \geqq 0)$ from this time, is meant the value of S discounted for n years times the probability of receiving S. If a person will win \$1000 in case he throws two heads in a single throw of two coins, the value of his expectation is $1000 \cdot \frac{1}{4} = \$250$; here $n = 0$. If a person aged 30 is to receive \$1000 at age 50, the value of his expectation is $1000\, v^{20} \cdot {}_{20}p_{30}$, where $v = \dfrac{1}{(1+i)}$. If two persons aged 25 and 30 are to receive \$1000 at the end of 20 years in case both are then living, the value of their expectation is $1000\, v^{20} \cdot {}_{20}p_{25} \cdot {}_{20}p_{30}$. If \$1000 is to be paid at the end of 20 years in case two persons aged 25 and 30 both live 19 years but both do not live 20 years, the value of the expectation that the money will be paid is $1000\, v^{20}({}_{19}p_{25} \cdot {}_{19}p_{30} - {}_{20}p_{25} \cdot {}_{20}p_{30})$.

EXERCISES

1. A person is to receive \$60 in case he throws a total of 7 in a single throw of two dice. Find the value of his expectation. Ans. \$10.

2. A person aged 30 is to receive \$1000 at the end of one year. Compute the value of his expectations. Use $i = .035$. Ans. \$958.04.

3. A person aged 30 is to receive \$1000 at the end of each year for 5 years. Compute the sum of the values of his expectations. Use $i = .035$.
Ans. \$4403.18.

4. A person aged 30 is to receive \$1000 at the end of each year for 5 years after he attains the age of 50. Find the sum of the values of his expectations. Use $i = .035$. Ans. \$1774.67.

5. If a person aged 30 dies within one year, his estate is to receive \$1000 at the end of the year. Find the value of the expectation. Use $i = .035$.
Ans. \$8.14.

77. Life annuities. A life annuity is a set of periodic payments, usually equal in value, during a term of years which begins at a specified age and continues during the whole or a part of the life of a person. This person is called an *annuitant*. As in annuities certain the term is from the beginning of the first period to the end of the last. When the term continues through the whole life of the person, the annuity is called a *whole life annuity:* when it ends at a stated time, even though the annuitant be still living, it is called a *temporary life annuity*. A *pure endowment* due in n years on the life of a person consists of a single payment at the end of n years in case the person is living at that time; that is, it is a temporary annuity having just one payment.

The classifications of whole life and temporary life annuities are analogous to those of annuities certain. One classification depends on when the term begins with respect to the age of the annuitant; it may begin at, after, or before this age. This gives rise respectively to *ordinary*, *deferred*, and *forborne* whole life and temporary life annuities. In what follows the word "ordinary" is omitted when there is no ambiguity in meaning. Another classification distinguishes whether the rent is paid at the end or the beginning of the rent period. This gives rise respectively to whole life and temporary life *annuities immediate*, and to whole life and temporary life *annuities due*. A third classification depends on the relative size of the rent payments; they may be equal, increasing, or

decreasing. The last two cases give rise to *increasing and decreasing* whole life and temporary life annuities.

The written contract between a company and an annuitant is called an *annuity policy.*

78. The value of a life annuity defined. If each of the l_{30} persons living at age 30, shown in a mortality table, holds a ten year temporary life annuity policy of annual rent 1 whose first payment is due at age 31, the amounts that will be due from these annuities at the end of 1, 2, $\cdots$, 10 years are, according to the mortality table, l_{31}, l_{32}, $\cdots$, l_{40} respectively. The sum of these amounts valued as of age 30, at an annual interest rate i, is $vl_{31} + v^2 l_{32} + \cdots + v^{10} l_{40}$ where $v = \dfrac{1}{(1+i)}\cdot$ The quotient,

$$V = \frac{vl_{31} + v^2 l_{32} + \cdots + v^{10} l_{40}}{l_{30}} (= \$7.95 \text{ at } i = .035)$$

gives what is called the value at age 30, at an interest rate i, of this temporary annuity as determined by the mortality table. The sum of the amounts due the annuitants valued as of age 40, at an interest rate i, is $(1+i)^9 l_{31} + (1+i)^8 l_{32} + \cdots + l_{40}$. The quotient

$$V = \frac{(1+i)^9 l_{31} + (1+i)^8 l_{32} + \cdots + l_{40}}{l_{40}} (= \$12.26 \text{ at } i = .035)$$

gives what is called the value at age 40, at an interest rate i, of this temporary annuity as determined by the mortality table. In other words the value at age 30 of this ten year temporary annuity is the amount that each of the l_{30} annuitants must pay to a company so that the total of these amounts with interest will be just enough to enable the company to pay 1 to each annuitant at the end of each year of the term of the annuity. Likewise, the value at age 40 of this annuity is the amount that each of the l_{40} persons would receive if the rent payments were not drawn when they are due, but were allowed to accumulate at compound interest to the time at age 40, and if the fund thus created were then equally divided among the l_{40} survivors. The values at the ages 30 and 40 of this temporary life annuity as just defined are included in the

Definition 1. *The value at age x of any life annuity of annual rent, R, is the quotient obtained by dividing l_x into the sum of the values at age x obtained by applying the compound interest formula to each sum of the set consisting of R times the number living, according to the mortality table, at the time of each rent payment during the term of the annuity.*

From this definition the value at age x of an ordinary or of a deferred life annuity of annual rent R may be viewed as the amount that each of the l_x annuitants must pay to a company so that the total of these amounts with interest will be just enough to enable the company to pay R to each annuitant living at the time of each rent payment during the term of the annuity. Likewise the value at age x of a forborne life annuity whose term begins n or more years prior to the time at age x, where n is the number of years in the term of the annuity, may be viewed as the amount that each of the l_x survivors would receive if the rent payments were not drawn when they are due but were allowed to accumulate at compound interest to the time at age x, and if the fund thus created were then equally divided among the l_x survivors.

When the annuity in definition 1 consists of a single sum payable at age $x + t$, the definition can be stated in the form:

The value, V, at age x of a one year temporary life annuity with annual rent R payable at age $x + t$ is given by

$$V = Rv^t \frac{l_{x+t}}{l_x} = R \frac{D_{x+t}}{D_x} \qquad (1_2)$$

$$= R \cdot v^t \cdot {}_tp_x \qquad \text{(when } t > 0\text{)}$$

where $D_x = v^x l_x$ *so that* $D_{x+t} = v^{x+t} l_{x+t}$. The second form of formula (1_2) is obtained from the first by multiplying numerator and denominator by v^x. When t is positive, V is the present or discounted value at age x of a sum R to be paid at the end of t years in case a person aged x survives t years; that is, when t is positive, V is the value at age x of a pure endowment due in t years. When t is negative, V is the amount at the end of t years of a sum R paid at age $x + t$ which is allowed to accumulate for t years, or to age x, as a pure endowment. The value at age x of a

pure endowment of 1 due in t years is denoted by ${}_tE_x$. It follows that

$$ {}_tE_x = v^t \frac{l_{x+t}}{l_x} = v^t \cdot {}_tp_x = \frac{D_{x+t}}{D_x} \qquad (1_3)$$

In finding the values at two or more ages of a given life annuity, use can be made of the following

Theorem III. *If V_{x+t} denotes the value at age $x + t$ of a life annuity, whose rent is payable annually, its value, V_x, at age x is given by*

$$V_x = v^t \frac{l_{x+t}}{l_x} V_{x+t} = \frac{D_{x+t}}{D_x} V_{x+t}.$$

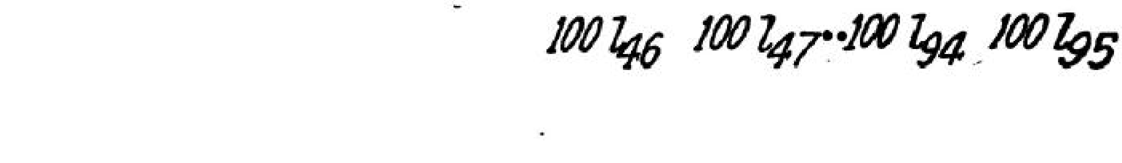
Proof: Since V_{x+t} is the value per person at age $x + t$, the aggregate value at age $x + t$ is $l_{x+t} V_{x+t}$. The value of this aggregate at age x is $v^t l_{x+t} V_{x+t}$ and the value per person at age x is $\frac{v^t l_{x+t} V_{x+t}}{l_x}$. This theorem for life annuities is the analogue of Theorem I, Art. 15, for annuities certain. When t is positive $\frac{D_{x+t}}{D_x}$ is a *discount factor;* when t is negative, it is an *accumulation factor*. It should be noted that Theorem III includes formula (1_2).

The symbol D_x is called a commutation symbol. Table XII gives the values at $i = .035$ of this and other commutation symbols. Computation of values of life annuities and life insurances are greatly facilitated by their use.

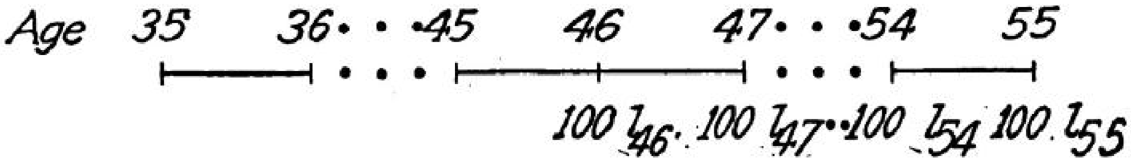
The value at age x of the annuity specified by any policy is called the *net single premium* at age x of the policy.

EXERCISES

1. Apply definition 1 to the set of sums below the line in the following diagram to write the expressions for the values at age 35, at age 45, at age 46, and at age 55 of a whole life annuity of annual rent $100 with first payment at age 46:



2. Apply definition 1 to the set of sums below the line in the following diagram to write the expressions for the values at age 35, at age 45, at age 46, and at age 55 of a ten year temporary life annuity of annual rent \$100 with first payment at age 46:



3. Apply definition 1 to write the expressions for the net single premiums at age 30 of the following life annuity policies, each having an annual rent of \$1000:

(*a*) Ordinary whole life annuity immediate,
(*b*) Ordinary whole life annuity due,
(*c*) Whole life annuity immediate deferred 10 years,
(*d*) Whole life annuity due deferred 10 years,
(*e*) Ordinary five year temporary life annuity immediate,
(*f*) Ordinary five year temporary life annuity due,
(*g*) Five year temporary life annuity immediate deferred 10 years,
(*h*) Five year temporary life annuity due deferred 10 years,
(*i*) Five year temporary life annuity d e forborne 5 years.

4. Compute the value at age 90 of an ordinary whole life annuity immediate of annual rent \$1000. Use definition 1 and $i = .035$. Ans. \$873.78.

5. Compute the value at age 25 of a pure endowment of \$1000 due in 25 years. Use $i = .035$. Ans. \$331.76.

6. Compute the present value, by use of the compound interest formula, of \$1000 due in 25 years. Use $i = .035$. Ans. \$423.15.

7. Compute the amount or the value at age 50 by use of formula (1_2) of \$1000 paid at age 25. Use $i = .035$. Ans. \$3014.23.

8. Compute the amount at age 50 by use of the compound interest formula of \$1000 paid at age 25. Use $i = .035$. Ans. \$2363.25.

79. Another definition of the value of a life annuity. The expressions given in Art. 78 for the values at the ages 30 and 40 of a ten year temporary life annuity immediate whose term begins at age 30 can be written in the forms:

$$V = v\frac{l_{31}}{l_{30}} + v^2\frac{l_{32}}{l_{30}} + \cdots + v^{10}\frac{l_{40}}{l_{30}} \quad \text{(at age 30), and}$$

$$V = v^{-9}\frac{l_{31}}{l_{40}} + v^{-8}\frac{l_{32}}{l_{40}} + \cdots + \frac{l_{40}}{l_{40}} \quad \text{(at age 40).}$$

These forms show that the value of this annuity at either age is the sum of the values obtained by applying formula (1_2) to each sum of the set consisting of 1 at the time of each rent payment during the term of the annuity. The value of the temporary life annuity defined in this way is included in the

Definition 2. *The value at age x of any life annuity of annual rent R is the sum of the values obtained by applying the formula*

$$V = Rv^t \frac{l_{x+t}}{l_x} = R\frac{D_{x+t}}{D_x}$$

to each sum of the set consisting of R at the time of each rent payment during the term of the annuity.

From this definition and the value of an expectation (Art. 76), the value at age x of an ordinary or of a deferred annuity of annual rent R is the sum of the values at age x of an annuitant's expectations of receiving the rent payments during the term of the annuity; or, it is the sum of the discounted values at age x, obtained by use of formula (1_2) when t is zero or positive, of the set of sums consisting of R at the time of each rent payment during the term of the annuity. Likewise, the value at age x of a forborne annuity, whose term begins n or more years prior to the time at age x, where n is the number of years in the term of the annuity, is the sum of the amounts at age x of the same set of sums when each is accumulated as a pure endowment, that is, by use of formula (1_2) when t is zero or negative.

The definitions in this and the preceding article evidently lead to the same value at age x of any life annuity. By definition 1, the compound interest formula is used to find the value at age x of a specified set of sums, each differing from the others, and this value is then divided by l_x. By definition 2, formula (1_2) is used to find the value at age x of a specified set of sums, each equal to R. In the above treatment this formula, (1_2), is obtained by use of definition 1. It can also be obtained by use of the definition of the value of an expectation (Art. 76), since, when $t \geq 0$, Rv^t is the value of R discounted for t years, and $\frac{l_{x+t}}{l_x}$ is the probability denoted by ${}_tp_x$. Other formulas for finding the value of a set of one or more

equal sums which are of fundamental importance in the theory of life annuities and of life insurances represent the values of expectations and can be written at once by one familiar with the elements of the theory of probability. Later in this chapter, analogous definitions are given for the value of any life insurance based on one life and for the values of joint life annuities and insurances.

EXERCISES

1. Apply definition 2 to the set of sums below the line in the following diagram to write the expressions for the values at age 35, at age 45, at age 46, and at age 55, of a whole life annuity of annual rent \$100 with first payment at age 46:

Age 35	36 ⋯ 45	46	47 ⋯ 94	95
		100	100 ⋯ 100	100

2. Apply definition 2 to the set of sums below the line in the following diagram to write the expressions for the values at age 35, at age 45, at age 46, and at age 55, of a ten year temporary life annuity of annual rent \$100 with first payment at age 46:

Age 35	36 ⋯ 45	46	47 ⋯ 54	55
		100	100 ⋯ 100	100

3. Same as Exercise 3, Art. 78, with "definition 1" replaced by "definition 2."

80. The value at age x of any whole life annuity of annual rent R. Let $x + t$ denote the age when the first rent payment of the annuity is made. By definition 1, Art. 78, the value, V, at age x is given by

$$V = R\frac{v^t l_{x+t} + v^{t+1} l_{x+t+1} + \text{etc.}}{l_x}$$

$$= R\frac{v^{x+t} l_{x+t} + v^{x+t+1} l_{x+t+1} + \text{etc.}}{v^x l_x}$$

$$= R\frac{D_{x+t} + D_{x+t+1} + \text{etc.}}{D_x}$$

$$= R\frac{N_{x+t}}{D_x} \qquad (1_1)$$

where the commutation symbol, N_x* is defined by

$$N_x = D_x + D_{x+1} + \text{etc., so that } N_{x+t} = D_{x+t} + D_{x+t+1} + \text{etc.}$$

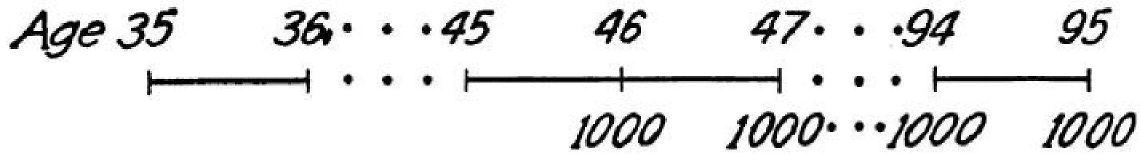
If the annual rent is 1, the values at age x of a whole life annuity immediate and of a whole life annuity due are denoted respectively by a_x, $\mathbf{a}_x$ when the terms begin at age x, and by ${}_{n_1|}a_x$, ${}_{n_1|}\mathbf{a}_x$ respectively when the terms begin at age $x + n_1$, that is, when the terms are deferred n_1 years. From formula (1_1) it follows that

$$a_x = \frac{N_{x+1}}{D_x}, \qquad \mathbf{a}_x = \frac{N_x}{D_x}$$

$$_{n_1|}a_x = \frac{N_{x+1+n_1}}{D_x} \qquad _{n_1|}\mathbf{a}_x = \frac{N_{x+n_1}}{D_x}$$

EXERCISES

1. Apply formula (1_1) to write the expressions for the values at age 35, at age 45, at age 46, and at age 55, of the whole life annuity of annual rent \$1000 represented by the following diagram:



2. Find the expressions for the values at age 35, at age 46, and at age 55 of the whole life annuity in Exercise 1 by applying Theorem III to the expression for its value at age 45.

3. Apply formula (1_1) to write the expressions for the net single premiums at age 25 of the following life annuity policies each having an annual rent of \$1000:

(*a*) Ordinary whole life annuity immediate,

(*b*) Ordinary whole life annuity due,

(*c*) Whole life annuity immediate deferred 10 years,

(*d*) Whole life annuity due deferred 10 years.

4. Use formula (1_1) at $i = .035$ to compute the value at age 25 of an ordinary whole life annuity immediate of annual rent \$1000. Ans. \$19441.73.

5. Find the net single premium at age 25 of a whole life annuity due of annual rent \$1000 deferred 25 years. Ans. \$4822.03.

* According to the definition adopted by the International Congress of Actuaries

$$N_x = D_{x+1} + D_{x+2} + \text{etc.}$$

The definition here given, however, is that generally used in America, the open bar N being used to distinguish it from that given for N_x.

6. The paid up value, $25000, of a life insurance policy is used to purchase a whole life annuity policy of annual rent R for a person aged 50. Find R if the first rent payment is made immediately. Use $i = .035$. Ans. $1720.62.

7. Solve Exercise 6 if the first instalment is made when the person aged 50 attains the age 65. Ans. $6427.15.

8. Derive formula (1_2) by use of definition 2, Art. 79.

9. Compute the value at age 67 of an ordinary whole life annuity immediate of annual rent $100. Use $i = .035$. Ans. $850.97.

10. At age 67 the complete expectation of life is 10 years. Compute the present value of the annuity certain $R = 100$, $n = 10$, $r = 1$ at $i = .035$ and compare with the result in Exercise 9. Ans. $831.66.

11. Use formula (1_1) to write the expressions for a_{25}, $\mathbf{a}_{25}$, ${}_{16|}a_{25}$, and ${}_{15|}\mathbf{a}_{25}$.

81. The value at age x of any n-year temporary life annuity of annual rent R. Let $x + t$ denote the age when the first rent payment of the annuity is made. By definition 1, Art. 78, the value, V, at age x is given by

$$\begin{aligned}
V &= R\frac{v^t l_{x+t} + v^{t+1} l_{x+t+1} + \cdots + v^{t+n-1} l_{x+t+n-1}}{l_x} \\
&= R\frac{v^{x+t} l_{x+t} + v^{x+t+1} l_{x+t+1} + \cdots + v^{x+t+n-1} l_{x+t+n-1}}{v^x l_x} \\
&= R\frac{D_{x+t} + D_{x+t+1} + \cdots + D_{x+t+n-1}}{D_x} \\
&= R\frac{N_{x+t} - N_{x+t+n}}{D_x} \qquad (1)
\end{aligned}$$

When the term of the temporary annuity continues throughout the life of the annuitant, $l_{x+t+n} = 0$ and hence $D_{x+t+n} = N_{x+t+n} = 0$. It follows that formula (1) includes formula (1_1). When $n = 1$, $N_{x+t} - N_{x+t+n} = D_{x+t}$ so that formula (1) includes formula (1_2).

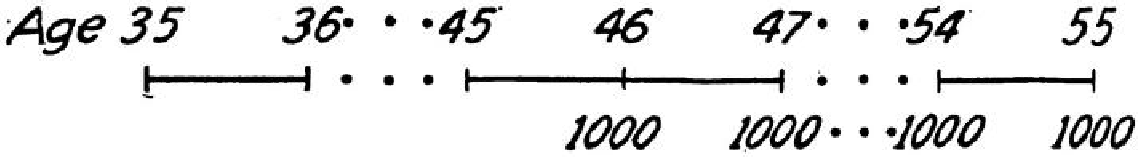
If the annual rent is 1, the values at age x of an n-year temporary life annuity immediate and of an n-year temporary life annuity due are denoted respectively by $a_{x\overline{n}|}$, $\mathbf{a}_{x\overline{n}|}$ when the terms begin at age x, and by ${}_{n_1|}a_{x\overline{n}|}$, and ${}_{n_1|}\mathbf{a}_{x\overline{n}|}$ respectively when the terms begin at age $x + n_1$, that is, when the terms are deferred n_1 years. From formula (1) it follows that

$$a_{x\overline{n}|} = \frac{N_{x+1} - N_{x+1+n}}{D_x}, \qquad \mathbf{a}_{x\overline{n}|} = \frac{N_x - N_{x+n}}{D_x}$$

$${}_{n_1|}a_{x\overline{n}|} = \frac{N_{x+1+n_1} - N_{x+1+n_1+n}}{D_x}, \quad {}_{n_1|}\mathbf{a}_{x\overline{n}|} = \frac{N_{x+n_1} - N_{x+n_1+n}}{D_x}$$

EXERCISES

1. Apply formula (1) to write the expressions for the values at age 35, at age 45, and at age 55, of the ten year temporary life annuity of annual rent $1000 represented by the following diagram:



2. Find the expressions for the values at age 35, at age 45, and at age 55, of the temporary annuity in Exercise 1 by applying Theorem III to the expression for its value at age 46.

3. Apply formula (1) to write the expressions for the net single premiums at age 25 of the following life annuity policies each having an annual rent of $1000:

(*a*) Ordinary ten year temporary life annuity immediate,
(*b*) Ordinary ten year temporary life annuity due,
(*c*) Ten year temporary life annuity immediate deferred 5 years,
(*d*) Ten year temporary life annuity due deferred 5 years,
(*e*) Ten year temporary life annuity due forborne 10 years.

4. Use formula (1) at $i = .035$ to compute the value at age 25 of an ordinary ten year temporary life annuity due of annual rent $100. Ans. $831.46.

5. Find the net single premium at age 25 of a ten year temporary life annuity immediate of annual rent $1000 deferred 10 years. Ans. $5158.91.

6. Compute the value at age 35 of a $1000 ten year life annuity due forborne 10 years. Ans. $12762.12.

7. The paid up value, $25000, of a life insurance policy is used to purchase a twenty year temporary life annuity of annual rent R for a person aged 50. Find R if the first rent payment is made immediately. Use $i = .035$.
Ans. $2006.90.

8. Solve Exercise 7 if the first instalment is made when the person aged 50 attains the age of 65. Ans. $6549.47.

9. Use formula (1_1) to derive formula (1) by resolving the n-year temporary life annuity whose first payment is made at age $x + t$ into the difference between two whole life annuities, one having its first payment at age $x + t$ and the other at age $x + t + n$.

10. Derive formula (1) by use of definition 2, Art. 78.

11. Use formula (1) to write the expressions for $a_{25:\overline{10}|}$, $\mathbf{a}_{25:\overline{10}|}$, $_{10|}a_{25:\overline{10}|}$, and $_{10|}\mathbf{a}_{25:\overline{10}|}$.

12. Prove Theorem III by use of formula 1.

82. Some important special cases of formula (1). By means of formula (1) the value of age x of any life annuity of annual rent R can be written at once. In this article important special cases are listed for the purpose of comparison.

	Whole Life Annuity		n-Year Temporary Life Annuity	
	Immediate	Due	Immediate	Due
Term begins at age x	$R\dfrac{N_{x+1}}{D_x}$	$R\dfrac{N_x}{D_x}$	$R\dfrac{N_{x+1}-N_{x+1+n}}{D_x}$	$R\dfrac{N_x-N_{x+n}}{D_n}$
Term deferred n_1 years	$R\dfrac{N_{x+1+n_1}}{D_x}$	$R\dfrac{N_{x+n_1}}{D_x}$	$R\dfrac{N_{x+1+n_1}-N_{x+1+n+n_1}}{D_x}$	$R\dfrac{N_{x+n_1}-N_{x+n+n_1}}{D_x}$
Term forborne n years	$R\dfrac{N_{x+1-n}}{D_x}$	$R\dfrac{N_{x-n}}{D_x}$	$R\dfrac{N_{x+n-1}}{D_x}$	$R\dfrac{N_{x-n}-N_x}{D_x}$

An n-year temporary life annuity whose term is forborne n years is usually called a *forborne life annuity*. In the application of forborne annuities it is frequently convenient to have x for the age at the beginning of the term rather than at the end. When this is done, the formula listed above for the value of a *forborne temporary life annuity due* becomes, upon replacing x by $x+n$,

$$V = R\frac{N_x - N_{x+n}}{D_{x+n}}.$$

When $R = 1$, the value of V is denoted by ${}_nu_x$. Hence

$$ {}_nu_x = \frac{N_x - N_{x+n}}{D_{x+n}} \tag{2}$$

When $n = 1$, ${}_nu_x$ is denoted by u_x. From formula (2) it follows that

$$u_x = \frac{D_x}{D_{x+1}}.$$

Values of u_x based on the American Experience Table at $3\frac{1}{2}\%$ are given in Table XIII.

EXERCISES

1. Write the expressions for the values of u_{20} and of ${}_{10}u_{20}$.
2. Compute the value of u_{20}.

83. Relations connecting the symbols for the values of life annuities. Important and interesting relations among the symbols for the values of life annuities can be readily determined. This can be done in various ways. One fruitful method is that of equating the value at age x of a given annuity to the sum of the values at this age of any set of annuities into which it can be resolved. By application of the method it is seen, for example, that

$\mathbf{a}_x = 1 + a_x$ (Whole life annuity due = 1 plus a whole life immediate.)

${}_{n_1|}\mathbf{a}_x = {}_{n_1|}a_x + {}_{n_1}E_x$ (Deferred whole life annuity due = deferred whole life annuity immediate plus a pure endowment.)

$\mathbf{a}_{x\overline{n}|} = a_{x\overline{n}|} + 1 - {}_nE_x$ (Temporary life annuity due = temporary life annuity immediate plus 1 minus a pure endowment.)

These relations can be written in the forms: $\mathbf{a}_x - a_x = 1$, ${}_{n_1|}\mathbf{a}_x - {}_{n_1|}a_x = {}_{n_1}E_x$ and $\mathbf{a}_{x\overline{n}|} - a_{x\overline{n}|} = 1 - {}_nE_x$. A more general relation which includes each of these is that which expresses the difference, Δ, between the values at age x of an n-year temporary life annuity of annual rent R whose first payment is made at age $x + k$, and a like annuity whose first payment is made at age $x + k + 1$. Since these two annuities have all their rent payments occurring at the same ages except that at age $x + k$ in the first and that at age $x + k + n$ in the second, it follows that

$$\Delta = R\,\frac{D_{x+k} - D_{x+k+n}}{D_x}. \tag{3}$$

Formula (3) is used in Art. 84 in finding the values of annuities whose rent payments are made more than once a year.

Theorem III, Art. 78, is also useful in determining relations. An illustration is the following: The value at age x of a whole life annuity immediate of annual rent 1 is denoted by a_x, and the value at age $x + 1$ of this same annuity is denoted by $\mathbf{a}_{x+1}$ or by $1 + a_{x+1}$. By the use of Theorem III it now follows that

$$a_x = v\,\frac{l_{x+1}}{l_x}\,(1 + a_{x+1}) = \frac{D_{x+1}}{D_x}\,(1 + a_{x+1})$$

This relation can be used to compute a table of values for a_x.

EXERCISES

1. Show that
   (a) $\mathbf{a}_{x\overline{n}|} = 1 + a_{x\overline{n-1}|}$
   (b) $a_x = a_{x\overline{n}|} + {}_n|a_x$
   (c) $a_{x+1} = u_x a_x - 1$
   (d) $\mathbf{a}_x = \dfrac{D_{x+1}}{D_x}\mathbf{a}_{x+1} + 1$
   (e) $\mathbf{a}_{x+1} = (\mathbf{a}_x - 1)u_x$
   (f) ${}_nu_x = ({}_{n-1}u_x + 1)u_{x+n-1}$
2. Interpret each of the relations in Exercise 1 verbally. (Note that, by Theorem III, u_x is an accumulation factor for one year and $\dfrac{D_{x+1}}{D_x}$ is a discount factor for one year.)

84. The value at age x of a life annuity whose rent is payable more than once a year. To find the value of a life annuity whose rent is payable more than once a year by direct application of the method given in Art. 78 would require a mortality table which shows the number living at each age, both fractional and integral, at which a rent payment is made. With such tables the computations would be quite tedious, especially if the number of payments each year were large. Satisfactory values for annuities payable m times a year, $(m > 1)$, can be readily found, however, by use of the values of annuities payable once a year and the method of simple interpolation. From formula (3), Art. 83, it follows that if each rent payment of an n-year temporary annuity whose annual rent is $\dfrac{R}{m}$ and whose first payment is made at age $x+k$ were made one year later, the value at age x of this annuity would be decreased by $\dfrac{\Delta}{m}$. By simple interpolation, the value of this annuity would be decreased by $\dfrac{1}{m}\cdot\dfrac{\Delta}{m}$ if each rent payment were made $\dfrac{1}{m}$th years later, by $\dfrac{2}{m}\cdot\dfrac{\Delta}{m}$ if each were made $\dfrac{2}{m}$th years later, and so on. An n-year temporary annuity due of total annual rent R payable in equal instalments m times a year whose term begins at age $x+k$ is made up of m, n-year temporary annuities each of rent $\dfrac{R}{m}$, pay-

able annually, whose rent payments begin at the ages, $x + k$, $x + k + \frac{1}{m}$, and so on. By use of formula (1) and simple interpolation, the values at age x of these annuities are given by

$$\frac{R}{m}\left(\frac{N_{x+k} - N_{x+k+n}}{D_x}\right)$$

$$\frac{R}{m}\left(\frac{N_{x+k} - N_{x+k+n}}{D_x}\right) - \frac{1}{m}\cdot\frac{\Delta}{m},$$

$$\frac{R}{m}\left(\frac{N_{x+k} - N_{x+k+n}}{D_x}\right) - \frac{2}{m}\cdot\frac{\Delta}{m},$$

. . . .

$$\frac{R}{m}\left(\frac{N_{x+k} - N_{x+k+n}}{D_x}\right) - \frac{m-1}{m}\cdot\frac{\Delta}{m}.$$

Summing these values by using the formula for the sum of an arithmetic progression, gives $R\left(\frac{N_{x+k} - N_{x+k+n}}{D_x}\right) - \frac{m-1}{2m}\Delta\cdot$ An analogous procedure for an n-year temporary annuity immediate whose term begins at age $x + k$ gives $R\left(\frac{N_{x+k} - N_{x+k+n}}{D_x}\right) - \frac{m+1}{2m}\Delta\cdot$ It follows that the values at age x for these m payment n-year temporary life annuities of total annual rent R whose *terms begin at age* $x + k$ are given by

$$\left.\begin{aligned} V &= R\left[\left(\frac{N_{x+k}-N_{x+k+n}}{D_x}\right) - \frac{m-1}{2m}\,\frac{D_{x+k}-D_{x+k+n}}{D_x}\right] \text{ for the annuity due,} \\ V &= R\left[\left(\frac{N_{x+k}-N_{x+k+n}}{D_x}\right) - \frac{m+1}{2m}\,\frac{D_{x+k}-D_{x+k+n}}{D_x}\right] \text{ for the annuity immediate.} \end{aligned}\right\} \quad (4)$$

The symbols for the values of life annuities of annual rent 1 payable m times a year are formed by writing the number m in parenthesis to the upper right of the symbols for the values of annuities of annual rent 1 payable once a year. For example, $a_x^{(m)}$ denotes the value at age x of a whole life annuity immediate of annual rent 1 payable m times a year. The expression for any one of these symbols can be written at once by use of formulas (4).

EXERCISES

1. Compute the value at age 30 of an ordinary whole life annuity immediate having a yearly rent of $1200 payable in equal instalments (*a*) annually, (*b*) semi-annually, (*c*) quarterly, and (*d*) monthly.

Ans. $22326.46; $22626.46; $22776.46; $22876.46.

2. Same as Exercise 1 except that the term is deferred 10 years.

Ans. $12789.68; $12984.10; $13081.30; $13146.11.

3. Same as Exercise 1 for a ten year temporary life annuity due.

4. Write the expressions for

(*a*) $a_x^{(m)}$ and $\mathbf{a}_x^{(m)}$

(*b*) $a_{x\overline{n}|}^{(m)}$ and $\mathbf{a}_{x\overline{n}|}^{(m)}$

(*c*) $_{n_1|}a_x^{(m)}$ and $_{n_1|}\mathbf{a}_{x\overline{n}|}$

5. Show that

(*a*) $a_x^{(m)} = a_x + \dfrac{m-1}{2m}$ and $\mathbf{a}_x^{(m)} = \mathbf{a}_x - \dfrac{m-1}{2m}$

(*b*) $a_{x\overline{n}|}^{(m)} = a_{x\overline{n}|} + \dfrac{m-1}{2m}(1 - {}_nE_x)$ and $\mathbf{a}_{x\overline{n}|}^{(m)} = \mathbf{a}_{x\overline{n}|} - \dfrac{m-1}{2m}(1 - {}_nE_x)$

(*c*) $_{n_1|}a_x^{(m)} = {}_{n_1|}a_x + \dfrac{m-1}{2m}\,{}_nE_x$ and $_{n_1|}\mathbf{a}_x^{(m)} = {}_{n_1|}\mathbf{a}_x - \dfrac{m-1}{2m}\,{}_nE_x$

85. Life Insurance. A life annuity policy is a contract which provides for the payment to the annuitant of stated amounts at periodic intervals during the whole or a part of his life. A life insurance policy on a person provides for the payment of a stated amount upon his death provided death takes place within a specified term of years. The amount to be paid is called the *face* of the policy or the insurance, the person on whose life it is taken is called the *insured*, and the person to whom the insurance is paid is called the *beneficiary*. The term during which a person is insured begins at a specified age and continues during the whole or a part of his life. When the term continues during the whole life of the person, the insurance is called *whole life insurance;* when it ends at a stated time, even though the insured be still living, it is called *term* * *life insurance.*

Both whole life and term insurances have classifications similar to those of annuities. One classification depends on when the term begins with respect to the age of the insured; it may begin

* The word "term" in term insurances takes the place of the word "temporary" in temporary annuities.

at, after, or before this age. This gives rise respectively to ordinary, deferred, and forborne whole life and term insurances. In referring to ordinary insurances it is customary to omit the word ordinary. The classification of life annuities into annuities immediate and annuities due does not carry over to life insurances since the insurance is paid to the beneficiary in a single payment, or in the form of an equivalent benefit, upon receipt of proof of death of the insured. In computing the net values of life insurances, however, it is usually assumed that payment of the insurance is made at the end of the year of death. Another classification depends on the relative size of the amounts to be paid. Some policies provide for amounts depending on the age at which death occurs. When these amounts increase with the age, the insurance is called *increasing insurance;* when they decrease, it is called *decreasing insurance.*

86. The value of life insurance defined. If each of the l_{30} persons living at age 30, shown in a mortality table, holds a ten year term life insurance policy of face value 1 whose term begins at age 30, the amounts that will be due their beneficiaries at the end of 1, 2, $\cdots$, 10 years are, according to the mortality table, d_{30}, d_{31}, $\cdots$, d_{39} respectively. The sum of these amounts valued as of age 30, at an interest rate i, is $vd_{30} + v^2d_{31} + \cdots + v^{10}d_{39}$ where $v = \dfrac{1}{(1+i)}$. The quotient

$$V = \frac{vd_{30} + v^2d_{31} + \cdots + v^{10}d_{39}}{l_{30}} \; (= .07129 \text{ at } i = .035)$$

gives what is called the value at age 30, at an interest rate i, of this term insurance, as determined by the mortality table. The sum of the amounts due the beneficiaries valued as of age 40 at an interest rate i is $(1+i)^9d_{30} + (1+i)^8d_{31} + \cdots + d_{39}$. The quotient

$$V = \frac{(1+i)^9d_{30} + (1+i)^8d_{31} + \cdots + d_{40}}{l_{40}} \; (= .1100 \text{ at } i = .035)$$

gives what is called the value at age 40, at an interest rate i, of this term insurance as determined by the mortality table. In

other words the value at age 30 of this ten year term insurance is the amount that each of the l_{30} insured must pay to a company so that the total of these amounts with interest will be just enough to enable the company to pay 1 to each beneficiary at the end of each year of the term of the insurance. Likewise, the value at age 40 of this insurance is the amount that each of the l_{40} persons would receive if the benefits were not drawn when they are due but were allowed to accumulate at compound interest to the time at age 40, and if the fund thus created were then equally divided among the l_{40} survivors. The values at the ages 30 and 40 of this term life insurance as just defined are included in the

Definition 1. *The value at age x of any life insurance of face value, F, is the quotient obtained by dividing l_x into the sum of the values at age x obtained by applying the compound interest formula to each sum in the set consisting of F times the number dying, according to the mortality table during each year of the term of the insurance.*

From this definition the value at age x of an ordinary or of a deferred life insurance of face value, F, may be viewed as the amount that each of the l_x insured must pay to a company so that the total of these amounts with interest will be just enough to enable the company to pay F to each beneficiary at the end of each year during the term of the insurance. Likewise the value at age x of a forborne life insurance, whose term begins n or more years prior to the time at age x, where n is the number of years in the term of the insurance, may be viewed as the amount that each of the l_x survivors would receive if the benefits were not drawn when they are due but were allowed to accumulate at compound interest to the time at age x, and if the fund thus created were then equally divided among the l_x survivors.

When the insurance in definition 1 is for a one year term beginning at the age $x + t - 1$, the definition can be stated in the form:

The value, V, at age x of an insurance of face value, F, for the year beginning at age $x + t - 1$ is given by

$$V = Fv^t \frac{d_{x+t-1}}{l_x} = F \frac{C_{x+t-1}}{D_x} \tag{5_2}$$

$$= Fv^t \cdot {}_{t-1|}q_x \qquad \text{(when } t > 0\text{)}$$

where $C_x = v^{x+1}d_x$, *so that* $C_{x+t-1} = v^{x+t}d_{x+t-1}$. The second form of formula (5_2) is obtained from the first by multiplying numerator and denominator by v^x. When t is positive, V is the present or discounted value at age x of a benefit F to be paid at the end of t years in case a person aged x dies during the year beginning at age $x + t - 1$. When t is negative, V is the amount at age x of the benefit F paid at age $x + t$, at the end of a one year term insurance, which is allowed to accumulate for t years in accordance with formula (5_2).

In finding the values at two or more ages of a given life insurance use can be made of the following

Theorem IV. *If V_{x+t} denotes the value at age $x + t$ of a life insurance, its value V_x at age x is given by*

$$V_x = v^t \frac{l_{x+t}}{l_x} V_{x+t} = \frac{D_{x+t}}{D_x} V_{x+t}.$$

The proof is the same as that for Theorem III, Art. 78.

In writing the equations needed for finding the unknowns in problems involving life annuities or life insurances use can be made of the

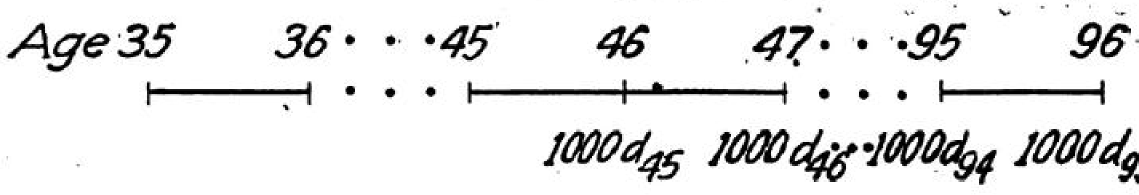
Theorem V. *If two life annuities, two life insurances, or a life annuity and a life insurance have equal values at age $x + t$, they have equal values at age x.*

This theorem is seen to be true by noting that Theorems III and IV have the same accumulation or discount factor, $\dfrac{D_{x+t}}{D_x}$. Theorem V is the analogue of Theorem II, Art. 16.

The value at age x of the insurance specified by any polic called the *net single premium* at this age, of the policy.

EXERCISES

1. Apply definition 1 to the set of sums below the line in the following diagram to write the expressions for the values at age 35, at age 45, and at age 55 of a whole life insurance of face value \$1000 whose term begins at age 45:



2. Apply definition 1 to the set of sums below the line in the following diagram to write the expressions for the values at age 35, at age 45, and at age 55 of a ten year term life insurance of face value \$1000, whose term begins at age 45:

Age 35   36 . . . 45   46   47 . . . 54   55

$1000d_{45}$ $1000d_{46}$ . . . $1000d_{53}$ $1000d_{54}$

3. Apply definition 1 to write the expressions for the net single premium at age 30 of the following \$1000 life insurance policies:

(a) Ordinary whole life,
(b) Whole life deferred 10 years,
(c) Ordinary ten year term,
(d) Ten year term deferred 10 years.

4. Compute the value at age 90 of an ordinary whole life insurance of face value \$1000. Use definition 1 and $i = .035$. Ans. \$936.64.

5. Compute the value at age 85 of a whole life insurance of face value \$1000 deferred 5 years. Use Theorem IV and the result found in Exercise 4.
Ans. \$121.78

6. Compute the value at age 60 by use of formula (5_2) of a \$1000 one year term insurance which matured at age 40; also the value at age 39. Ans. \$25.97; \$9.26.

87. Another definition of the value of life insurance. The expressions given in Art. 86 for the value at the ages 30 and 40 of a ten year term insurance whose term begins at age 30 can be written in the forms:

$$V = v\frac{d_{30}}{l_{30}} + v^2\frac{d_{31}}{l_{30}} + \cdots + v^{10}\frac{d_{39}}{l_{30}} \quad \text{(at age 30) and}$$

$$V = v^{-9}\frac{d_{30}}{l_{40}} + v^{-8}\frac{d_{30}}{l_{40}} + \cdots + \frac{d_{40}}{l_{40}} \quad \text{(at age 40).}$$

These forms show that the value of this insurance at either age is the sum of the values obtained by applying formula (5_2) to each sum of the set consisting of 1 payable at the end of each year of the term of the insurance. The value of the term life insurance defined in this way is included in the

Definition 2. *The value at age x of any life insurance of face value F is the sum of the values obtained by applying the formula*

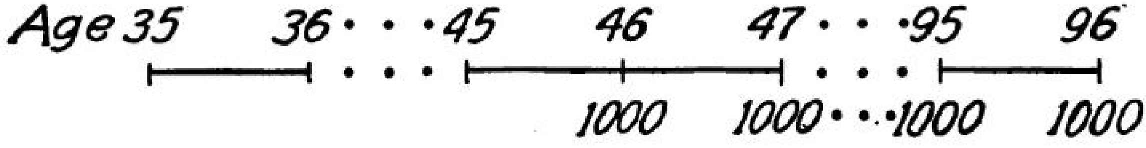
$$V = F\frac{v^t d_{x+t-1}}{l_x} = F\frac{C_{x+t-1}}{D_x}$$

to each sum in the set consisting of F payable at the end of each year of the term of the insurance.

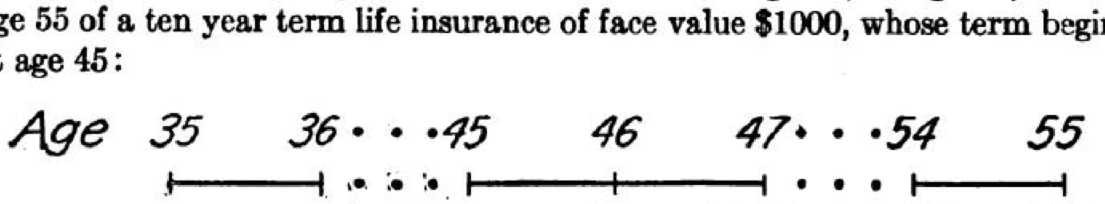
From this definition and the value of an expectation (Art. 76), the value at age x of an ordinary or of a deferred insurance of face value F is the sum of the values at age x (of the insured) of a beneficiary's expectations of receiving F at the end of each year during the term of the insurance; or it is the sum of the discounted values at age x, obtained by use of formula (5_2) when t is positive, of the set of sums consisting of F at the end of each year of the term of the insurance. Likewise, the value at age x of a forborne insurance whose term begins n or more years prior to the time at age x, where n is the number of years in the term of the insurance, is the sum of the amounts at age x of the same set of sums when each is accumulated by use of formula (5_2) when t is zero or negative.

EXERCISES

1. Apply definition 2 to the set of sums below the line in the following diagram to write the expressions for the values at age 35, at age 45, and at age 55 of a whole life insurance of face value \$1000 whose term begins at age 45:



2. Apply definition 2 to the set of sums below the line in the following diagram to write the expressions for the values at age 35, at age 45, and at age 55 of a ten year term life insurance of face value \$1000, whose term begins at age 45:



3. Same as Exercise 3, Art. 86, with "definition 1" replaced by "definition 2."

88. The value at age x of any whole life insurance of face value F. Let $x + k$ denote the age when the term of the insurance begins. By definition 1, Art. 86, the value V at age x is given by

$$V = F\,\frac{v^{k+1}d_{x+k} + v^{k+2}d_{x+k+1} + \text{etc.}}{l_x}$$

$$= F\,\frac{v^{x+k+1}d_{x+k} + v^{x+k+2}d_{x+k+1} + \text{etc.}}{v^x l_x}$$

$$= F\,\frac{C_{x+k} + C_{x+k+1} + \text{etc.}}{D_x}$$

$$= F\,\frac{M_{x+k}}{D_x} \tag{5_1}$$

where the commutation symbol, M_x, is defined by

$M_x = C_x + C_{x+1} +$ etc., so that $M_{x+k} = C_{x+k} + C_{x+k+1} + \cdots$

If the annual rent is 1, the value, at age x, of a whole life insurance is denoted by A_x when the term begins at age x, and by ${}_{n_1|}A_x$ when the term begins at age $x + n_1$, that is, when the term is deferred n_1 years. From formula (5_1) it follows that

$$A_x = \frac{M_x}{D_x}, \qquad {}_{n_1|}A_x = \frac{M_{x+n_1}}{D_x}$$

EXERCISES

1. Apply formula (5_1) to write the expressions for the values at age 35, at age 45, and at age 55, of the whole life insurance of face value \$1000 whose term begins at age 45, represented by the following diagram:

Age 35 · 36 · · · 45 46 47 · · · 95 96

1000 1000 · · · 1000 1000

2. Find the expressions for the values at age 35 and at age 55 of the whole life insurance in Exercise 1 by applying Theorem IV to the expression for its value at age 45.

3. Apply formula (5_1) to write the expressions for the net single premiums at age 25 of the following \$1000 life insurance policies:

(*a*) Ordinary whole life,

(*b*) Whole life deferred 10 years.

4. Use formula (5_1) at $i = .035$ to compute the value at age 25 of a \$1000 ordinary whole life insurance. Ans. \$308.73.

5. Find the net single premium at age 25 of a \$1000 whole life insurance deferred 10 years. Ans. \$241.41.

6. Derive formula (5_1) by use of definition 2, Art. 85.

7. Use formula (5_1) to write the expressions for the values of A_{25} and ${}_{10|}A_{25}$.

89. The value at age *x* of any *n*-year term insurance of face value *F*. Let $x + k$ denote the age when the term of the insurance begins. By definition 1, Art. 86, the value, V, at age x is given by

$$
\begin{aligned}
V &= F\frac{v^{k+1}d_{x+k} + v^{k+2}d_{x+k+1} + \cdots + v^{k+n}d_{x+k+n-1}}{l_x} \\
&= F\frac{v^{x+k+1}d_{x+k} + v^{x+k+2}d_{x+k+1} + \cdots + v^{x+k+n}d_{x+k+n-1}}{v^x l_x} \\
&= F\frac{C_{x+k} + C_{x+k+1} + \cdots + C_{x+k+n-1}}{D_x} \\
&= F\frac{M_{x+k} - M_{x+k+n}}{D_x} \qquad (5)
\end{aligned}
$$

When the term of the insurance continues throughout the life of the insured, $d_{x+k+n} = 0$ and hence $C_{x+k+n} = M_{x+k+n} = 0$. It follows that formula (5) includes formula (5_1). When $k = t - 1$, and $n = 1$, $M_{x+k} - M_{x+k+n} = C_{x+t-1}$, so that formula (5) includes formula (5_2). Formula (5) should be thoroughly mastered.

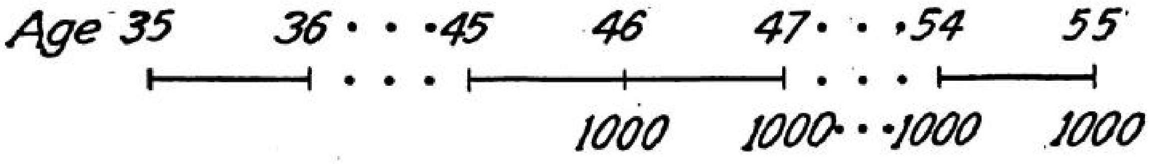
It should be especially noted that in formula (1), $x + t$ is the age when the first rent payment is made, while in formula (5), $x + k$ is the age when the term begins.

If the annual rent is 1, the value at age x of an n-year term insurance is denoted by $A^1_{x\overline{n|}}$ when the term begins at age x, and by ${}_{n_1|}A^1_{x\overline{n|}}$ when the term begins at age $x + n_1$. From formula (5) it follows that

$$A^1_{x\overline{n|}} = \frac{M_x - M_{x+n}}{D_x}, \quad {}_{n_1|}A^1_{x\overline{n|}} = \frac{M_{x+n_1} - M_{x+n_1+n}}{D_x}$$

EXERCISES

1. Apply formula (5) to write the expressions for the values at age 35, at age 45, and at age 55, of the ten year term life insurance of face value \$1000 whose term begins at age 45 represented by the following diagram:



2. Find the expressions for the values at age 35 and at age 55 of the term insurance in Exercise 1 by applying Theorem IV to the expression for its value at age 45.

3. Apply formula (5) to write the expressions for the net single premiums at age 25 of the following \$1000 life insurance policies:

(*a*) Ordinary ten year term,

(*b*) Ten year term deferred 10 years.

4. Use formula (5) at $i = .035$ to compute the value at age 25 of a \$1000 ordinary ten year term insurance. Ans. \$67.32.

5. Find the net single premium at age 25 of a \$1000 ten year term insurance deferred 5 years. Ans. \$57.60.

6. Derive formula (5) by use of definition 2, Art. 87.

7. Use formula (5) to write the expressions for the values of $A^{1}_{25\,\overline{10}|}$ and ${}_{10|}A^{1}_{25\,\overline{10}|}$.

8. Use formula (5_1) to derive formula (5) by resolving the n-year term insurance whose term begins at age $x + k$ into the difference between two whole life insurances whose terms begin at the ages $x + k$ and $x + k + n$.

9. Prove Theorem IV by use of formula (5).

90. Some important special cases of formula (5). By means of formula (5) the value at age x of any life insurance of face value F can be written at once. In this article important special cases are listed for the purpose of comparison.

	Whole Life Insurance	n-Year Term Life Insurance
Term begins at age x	$F\dfrac{M_x}{D_x}$	$F\dfrac{M_x - M_{x+n}}{D_x}$
Term deferred n_1 years	$F\dfrac{M_{x+n_1}}{D_x}$	$F\dfrac{M_{x+n_1} - M_{x+n_1+n}}{D_x}$
Term forborne n years	$F\dfrac{M_{x+n}}{D_x}$	$F\dfrac{M_{x-n} - M_x}{D_x}$

An n-year term life insurance whose term is forborne n years is usually called a *forborne life insurance*. In the application of forborne insurance it is frequently convenient to have x for the age at the beginning of the term rather than at the end. When this is done, the formula listed above for the value of a forborne n-year term insurance becomes, upon replacing x by $x + n$,

$$V = F\frac{M_x - M_{x+n}}{D_{x+n}}$$

When $F = 1$, the value of V is denoted by ${}_nk_x$. Hence

$$ {}_nk_x = \frac{M_x - M_{x+n}}{D_{x+n}} \tag{6}$$

When $n = 1$, ${}_nk_x$ is denoted by k_x. From formula (6) it follows that

$$k_x = \frac{C_x}{D_{x+1}}$$

Values of k_x based on the American Experience Table at $3\frac{1}{2}\%$ are given in Table XIII.

EXERCISES

1. Write the expressions for the values of k_{20} and ${}_{10}k_{20}$.
2. Compute the value of k_{20}.

91. Life annuities and life insurances combined; n-year endowment insurance. In the preceding articles in this chapter the values at age x of life annuities and of life insurances have been determined. The value at age x of any combination of a life annuity and of a life insurance may be found by summing the separate values. A common form among such combinations is an *n-year endowment insurance,* which combines an n-year term insurance with a pure endowment due in n years. If the face value of each of these is F, the value at age x of the term insurance is $F\frac{M_x - M_{x+n}}{D_x}$, and that of the pure endowment is $F\frac{D_{x+n}}{D_x}$. It follows that the value, V, at age x of an n-year endowment insurance of face value, F, is given by

$$V = F\frac{M_x - M_{x+n} + D_{x+n}}{D_x} \tag{7}$$

When $F = 1$, the value of V is denoted by $A_{x\overline{n}|}$, so that

$$A_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

The values of other combinations of life annuities and life insurances can be written at once by use of formulas (1) and (5).

The value of V for an n-year endowment insurance policy is called the *net single premium* of the policy. A modified form of an n-year endowment insurance policy specifies the age at maturity. For example, an *endowment at age* 65 issued at age 22 is the same as a 43-year endowment insurance.

EXERCISES

1. Apply formulas (5) and (1_2) to write the expressions for the values at age 35, and at age 45, of the \$1000 twenty year endowment insurance whose term begins at age 45, represented by the following diagram:

Age	35	36	⋯	45	46	47	⋯	64	⋯	65
Insurance					1000	1000		1000	⋯	1000
Pure endowment										1000

2. Compute the values at $i = .035$ of the expressions found in Exercise 1.
Ans. \$360.03; \$560.23.

3. Compute the net single premium at age 25 of a \$1000 ten year endowment insurance. Use $i = .035$.

4. If the net single premium of a 20-year endowment insurance policy issued at age 30 is \$1000, what is the face value of the policy? Use $i = .035$.
Ans. \$1855.90.

5. Write the expression for the value at age 30 of a \$1000 ordinary 35-year term insurance combined with a \$1000 whole life annuity immediate deferred 35 years. Draw a diagram similar to that in Exercise 1. Compute the value of this expression at $i = .035$. Ans. \$1641.64.

6. Find the net single premium at age 27 of a \$1000 endowment at age 60.
Ans. \$398.73.

92. Relations connecting the symbols denoting the values of (1) life insurances, (2) life annuities and life insurances. The methods used in Art. 83 to find relations among life annuity symbols lead to similar relations among life insurance symbols. For example, the relation among the insurance symbols which corresponds to the formula at the end of Art. 83 can be obtained as

follows: The value at age x of a whole life insurance of face value 1 is denoted by A_x. The value of this same insurance at age $x + 1$ is $\frac{d_x}{l_{x+1}} + A_{x+1}$. This is seen at once by resolving the insurance into a one year term insurance whose term begins at age x and a whole life insurance whose term begins at age $x + 1$; A_{x+1} denotes the value at age $x + 1$ of the whole life component and, by formula (5_2), $\frac{d_x}{l_{x+1}}$ is the value at age $x + 1$ of the one year term insurance. Hence by Theorem IV

$$\begin{aligned} A_x &= v\frac{l_{x+1}}{l_x}\left(\frac{d_x}{l_{x+1}} + A_{x+1}\right) \\ &= v(q_x + p_x A_{x+1}) \end{aligned}$$

This relation can be used to compute a table of values for A_x.

The values of life insurances are based on the d_x column in a mortality table. The values of life annuities are based on the l_x column. By use of the relation $d_x = l_x - l_{x+1}$, it can be readily proved that $C_x = vD_x - D_{x+1}$ and $M_x = vN_x - N_{x+1}$. The proofs are as follows:

$$\begin{aligned} C_x &= v^{x+1}d_x \\ &= v^{x+1}(l_x - l_{x+1}) \\ &= vD_x - D_{x+1} \\ M_x &= C_x + C_{x+1} + \cdots + C_{95} \\ &= (vD_x - D_{x+1}) + (vD_{x+1} - D_{x+2}) + \cdots + (vD_{95} - D_{96}) \\ &= v(D_x + D_{x+1} + \cdots + D_{95}) - (D_{x+1} + D_{x+2} + \cdots + D_{96}) \\ &= vN_x - N_{x+1} \end{aligned}$$

From this value of M_x it follows that the value at age x of any life insurance can be expressed in terms of values of life annuities. For example:

$$\begin{aligned} A_x &= \frac{M_x}{D_x} \\ &= \frac{vN_x - N_{x+1}}{D_x} \\ &= v\mathbf{a}_x - a_x = v(1 + a_x) - a_x \\ &= 1 - d(1 + a_x) \qquad \text{(where } v = 1 - d\text{)} \\ &= 1 - d\mathbf{a}_x \end{aligned}$$

EXERCISES

1. Show that

(a) $A^{1}_{x\overline{n}|} = v\mathrm{a}_{x\overline{n}|} - a_{x\overline{n}|}$ (c) $A_x = A^{1}_{x\overline{n}|} + {}_{n|}A_x$

(b) $A^{1}_{x\overline{n}|} = v(q_x + p_x A^{1}_{x+1:\overline{n-1}|})$ (d) $A_{x+1} = u_x A_x - k_x$

2. Interpret each of the relations in Exercise 1 verbally. Also interpret the relation, $A_x = 1 - d(1 + a_x)$, verbally.

93. The value of a set of life insurances or a set of life annuities. Increasing and decreasing insurances; increasing and decreasing annuities. The value at age x of any set of life annuities or of life insurances can be found by summing the values of the separate sets. The values of some sets can be found more easily, however, by basing the computations on simplified forms of the expressions for the sum of the values of the given sets. Illustrations are afforded by sets of n life insurances each of face value F, or of n life annuities each of rent R, having terms of n, $n - 1$, $\cdots$, 1 years respectively, all of which either begin at age x or end at age $x + n$. When the terms all begin at the same age x, the sum forms a decreasing insurance or a decreasing annuity; when the terms all end at the same age $x + n$, the sum forms an increasing insurance or an increasing annuity. In the increasing insurance the face value increases by F each year; in the decreasing insurance it decreases by F each year. Likewise in the increasing annuity the rent increases by R each year; in the decreasing annuity it decreases by R each year.

The value, V, at age x of an increasing insurance whose face value begins with F and increases by F each year, and whose term begins at age x can be found by the following process: An increasing insurance of F whose term begins at age x and ends at age $x + n$ is the sum of n term insurances whose terms begin at the ages x, $x + 1$, $\cdots$, $x + n - 1$ respectively and end at age $x + n$. Hence

$$V = F\left(\frac{M_x - M_{x+n}}{D_x} + \frac{M_{x+1} - M_{x+n}}{D_x} + \cdots + \frac{M_{x+n-1} - M_{x+n}}{D_x}\right)$$

$$= F\frac{M_x + M_{x+1} + \cdots + M_{x+n-1} - nM_{x+n}}{D_x}$$

$$= F\frac{R_x - R_{x+n} - nM_{x+n}}{D_x} \text{ where } R_x = M_x + M_{x+1} + \cdots, \quad (8)$$

When $F = 1$, V is denoted by $(IA^{1}_{x\overline{n}|})$; in this case formula (8) becomes

$$(IA^{1}_{x\overline{n}|}) = \frac{R_x - R_{x+n} - nM_{x+n}}{D_x} \tag{8$_1$}$$

The expression for the value at age x of an increasing annuity whose first payment is at age x, can be found in an analogous manner. If the rent begins with R and increases by R each year, the value, V, is given by

$$V = R\left(\frac{N_x - N_{x+n}}{D_x} + \frac{N_{x+1} - N_{x+n}}{D_x} + \frac{N_{x+2} - N_{x+n}}{D_x} + \cdots + \frac{N_{x+n-1} - N_{x+n}}{D_x}\right)$$

$$= R\frac{N_x + N_{x+1} + \cdots + N_{x+n-1} - nN_{x+n}}{D_x}$$

$$= R\frac{S_x - S_{x+n} - nN_{x+n}}{D_x} \tag{9}$$

where $S_x = N_x + N_{x+1} + N_{x+2} + \cdots$

When $R = 1$, V is denoted by $(I\mathrm{a}_{x\overline{n}|})$; in this case formula (9) becomes

$$(I\mathrm{a}_{x\overline{n}|}) = \frac{S_x - S_{x+n} - nN_{x+n}}{D_x} \tag{9$_1$}$$

Values of the commutation symbols R_x and S_x are not included in the tables at the end of this book.* Their values can be found by use of the M- and N-columns.

EXERCISES

1. Compute the value at age 35 of an increasing insurance of \$1000 issued at age 35 for a term of 10 years. Ans. \$411.45.

2. Same as Exercise 1 for an increasing annuity. Ans. \$42518.72.

3. Show that the value at age x of a whole life increasing insurance of F issued at age x is given by

$$V = F\frac{R_x}{D_x}$$

4. Compute the value at age 25 of the insurance in Exercise 1.

Ans. \$268.06.

* The values of R_x and S_x at .03, 035, and .04, are given in Glover's Tables of Applied Mathematics.

94. Joint life annuities. In a single life annuity the rent payments continue as long during the term as the single life survives. In a joint life annuity the rent payments continue as long during the term as *all* the lives survive; when any life fails, the payments cease. Joint life annuities are classified in a manner analogous to single life annuities. In the next three articles formulas are developed for the values of joint life annuities involving two lives, which are analogous to those for single lives. The methods, however, are applicable to any number of lives. In Art. 104 an approximate method for computing values of joint life annuities is presented.

95. The value of a joint life annuity defined. If each of the l_{28} persons living at age 28, shown in the mortality table, is paired with each of the l_{32} persons living at age 32, there are $l_{28}l_{32}$ such pairs. Of these pairs $l_{29}l_{33}$ survive one year, $l_{30}l_{34}$ survive two years, and so on. If each of the $l_{28}l_{32}$ pairs of persons of ages 28, 32 holds a ten year temporary joint life annuity policy of annual rent 1, the amounts that will be received on these policies at the end of 1, 2, $\cdots$, 10 years are $l_{29}l_{33}$, $l_{30}l_{34}$, $\cdots$ $l_{38}l_{42}$ respectively. The sum of these amounts, valued as of the ages 28, 32 at an annual interest rate i, is $vl_{29}l_{33} + v^2l_{30}l_{34} + \cdots + v^{10}l_{38}l_{42}$ where $v = \dfrac{1}{(1+i)}$. The quotient

$$V = \frac{vl_{29}l_{33} + v^2l_{30}l_{34} + \cdots + v^{10}l_{38}l_{42}}{l_{28}l_{32}}$$

gives what is called the value at the ages 28, 32, at an interest rate i of this temporary joint life annuity as determined by the mortality table. The sum of the amounts due the pairs of annuitants valued as of ages 38, 42 at an interest rate i, is $(1+i)^9l_{29}l_{33} + (1+i)^8l_{30}l_{34} + \cdots + l_{38}l_{42}$. The quotient

$$V = \frac{(1+i)^9l_{29}l_{33} + (1+i)^8l_{30}l_{34} + \cdots + l_{38}l_{42}}{l_{38}l_{42}}$$

gives what is called the value at the ages 38, 42, at an interest rate i of this temporary joint life annuity as determined by the mortality table. In other words, the value at ages 28, 32 of this

ten year temporary joint life annuity is the amount that each of the $l_{28}l_{32}$ pairs of annuitants must pay to a company so that the total of these amounts with interest will be just enough to enable the company to pay 1 to each pair of annuitants at the end of each year of the term of the annuity. Likewise, the value at ages 38, 42 of this joint life annuity is the amount that each of the $l_{38}l_{42}$ pairs of persons would receive if the rent payments were not drawn when they are due but were allowed to accumulate at compound interest to the time at the ages 38, 42, and if the fund thus created were then equally divided among the $l_{38}l_{42}$ pairs of survivors. The values at the pairs of ages 28, 32 and 38, 42 of this temporary joint life annuity as just defined are included in the

Definition 1. *The value at the ages x, y of any joint life annuity of annual rent, R, is the quotient obtained by dividing l_xl_y into the sum of the values obtained by applying the compound interest formula to each sum of the set consisting of R times the number of pairs living, according to the mortality table, at the time of each rent payment during the term of the annuity.*

From this definition the value at the ages x, y of an ordinary or of a deferred joint life annuity of annual rent R may be viewed as the amount that each of the l_xl_y pairs of annuitants must pay to a company so that the total of these amounts with interest will be just enough to enable the company to pay R to each pair of annuitants living at the time of each rent payment during the term of the annuity. Likewise, the value at the ages x, y of a forborne joint life annuity whose term begins n or more years prior to the ages x, y, where n is the number of years in the term of the annuity, may be viewed as the amount that each of the l_xl_y pairs of survivors would receive if the rent payments were not drawn when they are due but were allowed to accumulate at compound interest to the time at the ages x, y and if the fund thus created were then equally divided among the l_xl_y pairs of survivors.

When the joint life annuity in definition 1 consists of a single sum payable at the ages $x + t$, $y + t$, the definition can be stated in the form:

The value, V, at the ages x, y of a one year temporary joint life annuity with annual rent R payable at the ages $x + t$, $y + t$ is given by

$$V = Rv^t \frac{l_{x+t}l_{y+t}}{l_x l_y} = R\frac{D_{x+t:y+t}}{D_{xy}} \tag{10$_2$}$$

$$= Rv^t \cdot {}_tp_{xy} \qquad \text{(when } t > 0\text{)}$$

where $D_{xy} = v^{\frac{x+y}{2}} l_x l_y$ *so that* $D_{x+t:y+t} = v^{\frac{x+y}{2}+t} l_{x+t}l_{y+t}$. The second form of formula (10_2) is obtained from the first by multiplying numerator and denominator by $v^{\frac{x+y}{2}}$. When t is positive, V is the present or discounted value at the ages x, y of a sum R to be paid at the end of t years in case two persons aged x and y survive t years; that is, when t is positive, V is the value at the ages x, y of a joint life pure endowment due in t years. When t is negative, V is the amount at the end of t years of a sum R paid at the ages $x + t$, $y + t$ which is allowed to accumulate for t years, or to the ages x, y, as a joint life pure endowment. The value at the ages x, y of a joint life pure endowment of 1 due in t years is denoted by ${}_tE_{xy}$. It follows that

$${}_tE_{xy} = v^t \frac{l_{x+t}l_{y+t}}{l_x l_y} = v^t \cdot {}_tp_{xy} = \frac{D_{x+t:y+t}}{D_{xy}} \tag{10$_3$}$$

In finding the values at two or more pairs of ages of a joint life annuity use can be made of the following

Theorem VI. If $V_{x+t:y+t}$ *denotes the value at the ages* $x + t$, $y + t$ *of a joint life annuity, its value* V_{xy} *at the ages* x, y *is given by*

$$V_{xy} = v^t \frac{l_{x+t}l_{y+t}}{l_x l_y} V_{x+t:y+t} = \frac{D_{x+t:y+t}}{D_{xy}} V_{x+t:y+t}$$

This theorem is an extension of Theorem III. The proof is left as an exercise. When t is positive, $\frac{D_{x+t:y+t}}{D_{xy}}$ is a *discount factor*; when t is negative, it is an *accumulation factor*. It should be noted that Theorem VI includes formula (10_2).

The value at the ages x, y of the joint life annuity specified by any policy is called the *net single premium* of the policy.

In the following articles $l_x l_y$, $l_{x+1}l_{y+1}$, and so on will often be written l_{xy}, $l_{x+1:y+1}$, and so on for brevity.

EXERCISES

1. Apply definition 1 to the set of sums below the line in the following diagram to write the expressions for the values at the ages 35, 40, at the ages 45, 50, and at the ages 46, 51, of a joint whole life annuity of annual rent 100 with first payment at the ages 46, 51:

$$\text{Ages}\begin{cases}35 & 36\cdots45 & 46 & 47\cdots89 & 90\\ 40 & 41\cdots50 & 51 & 52\cdots94 & 95\end{cases}$$

$$100\,l_{46}l_{51}\quad 100\,l_{47}l_{52}\cdots100\,l_{89}l_{94}\quad 100\,l_{90}l_{95}$$

2. Compute the value at the ages 25, 30 of a joint life pure endowment of \$1000 due in 25 years. Use $i = .035$. Ans. \$250.69.

3. Use definition 1 at $i = .035$ to compute the value at the ages 90, 92 of an ordinary joint whole life annuity immediate of annual rent \$1000.

Ans. \$217.06.

96. Another definition of a joint life annuity. The expressions given in Art. 95 for the values at the pairs of ages 28, 32 and 38, 42 of a ten year temporary joint life annuity immediate whose term begins at the ages 28, 32 can be written in the forms:

$$V = v\frac{l_{29}l_{33}}{l_{28}l_{32}} + v^2\frac{l_{30}l_{34}}{l_{28}l_{32}} + \cdots + v^{10}\frac{l_{38}l_{42}}{l_{28}l_{42}} \quad \text{(at ages 28, 32) and}$$

$$V = v^{-9}\frac{l_{29}l_{33}}{l_{38}l_{42}} + v^{-8}\frac{l_{30}l_{34}}{l_{38}l_{42}} + \cdots + \frac{l_{38}l_{42}}{l_{38}l_{42}} \quad \text{(at ages 38, 42)}$$

These forms show that the value of this annuity at either pair of ages is the sum of the values obtained by applying formula (10_2) to each sum of the set consisting of 1 at the time of each rent payment during the term of the annuity. The value of the temporary joint life annuity defined in this way is included in the

Definition 2. *The value at the ages x, y of any joint life annuity of annual rent R is the sum of the values obtained by applying the formula*

$$V = Rv^t\frac{l_{x+t}l_{y+t}}{l_xl_y} = R\frac{D_{x+t:y+t}}{D_{xy}}$$

to each sum of the set consisting of R at the time of each rent payment during the term of the annuity.

From this definition and the value of an expectation (Art. 76), the value at the ages x, y of an ordinary or of a deferred joint life annuity of annual rent R is the sum of the values at the ages x, y of the expectations of a pair of annuitants of receiving the rent payments during the term of the annuity; or it is the sum of the discounted values at the ages x, y, obtained by use of formula (10_2) when t is zero or positive of the set of sums consisting of R at the time of each rent payment during the term of the annuity. Likewise, the value at the ages x, y of a forborne joint life annuity, whose term begins n or more years prior to the time at the ages x, y, where n is the number of years in the term of the annuity, is the sum of the amounts at the ages x, y of the same set of sums when each is accumulated as a joint life pure endowment, that is, by use of formula (10_2) when t is zero or negative.

EXERCISE

Apply definition 2 to the set of sums below the line in the following diagram to write the expressions for the values at the ages 35, 40, at the ages 45, 50, and at the ages 55, 60, of a 10 year temporary joint life annuity of annual rent 100 with first payment at the ages 46, 51:

$$\begin{array}{lcccccc} \text{Ages}\begin{cases} \\ \\ \end{cases} & \begin{matrix}35\\40\end{matrix} & \begin{matrix}36\cdots45\\41\cdots50\end{matrix} & \begin{matrix}46\\51\end{matrix} & \begin{matrix}47\cdots54\\52\cdots59\end{matrix} & \begin{matrix}55\\60\end{matrix} \\ & & & 100 & 100\cdots100 & 100 \end{array}$$

97. The value at the ages x, y of any joint whole life annuity of annual rent R. Let $x+t$, $y+t$ denote the ages when the first rent payment is made. By definition 1, Art. 95, the value, V, at the ages x, y is given by

$$V = R\frac{v^t l_{x+t:y+t} + v^{t+1} l_{x+t+1:y+t+1} + \text{etc.}}{l_{xy}}$$

$$= R\frac{v^{\frac{x+y}{2}+t} l_{x+t:y+t} + v^{\frac{x+y}{2}+t+1} l_{x+t+1:y+t+1} + \text{etc.}}{v^{\frac{x+y}{2}} l_{xy}}$$

$$= R\frac{D_{x+t:y+t} + D_{x+t+1:y+t+1} + \text{etc.}}{D_{xy}}$$

$$= R\frac{N_{x+t:y+t}}{D_{xy}} \qquad (10_1)$$

where the commutation symbol N_{xy} is defined by

$$N_{xy} = D_{xy} + D_{x+1:y+1} + \text{etc.},$$

so that $$N_{x+t:y+t} = D_{x+t:y+t} + D_{x+t+1:y+t+1} + \text{etc.}$$

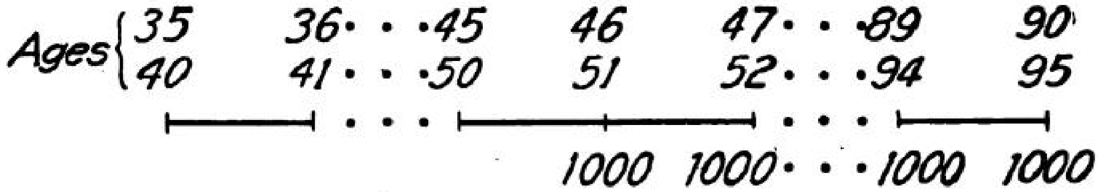
The symbols for the values of the joint whole life annuities of annual rent 1 are analogous to those for single lives given in Art. 80. From formula (10_1) it follows that

$$a_{xy} = \frac{N_{x+1:y+1}}{D_{xy}}, \qquad \mathbf{a}_{xy} = \frac{N_{xy}}{D_{xy}}$$

$$_{n_1|}a_{xy} = \frac{N_{x+1+n_1:y+1+n_1}}{D_{xy}} \qquad _{n_1|}\mathbf{a}_{xy} = \frac{N_{x+n_1:y+n_1}}{D_{xy}}$$

EXERCISES

1. Apply formula (10_1) to write the expressions for the values at the ages 35, 40, and at the ages 46, 51, of the joint whole life annuity of annual rent 1000 represented by the following diagram:



2. Find the expression for the value at the ages 35, 40 of the joint whole life annuity in Exercise 1 by applying Theorem VI to its value at the ages 46, 51.

3. Apply formula (10_1) to write the expressions for the net single premiums at the ages 25, 35 of the following joint life annuity policies each having an annual rent of \$1000:

(a) Ordinary joint whole life annuity immediate,
(b) Ordinary joint whole life annuity due,
(c) Joint whole life annuity immediate deferred 10 years,
(d) Joint whole life annuity due deferred 10 years.

4. Use formula (10_1) to write the expressions for $a_{25:35}$, $\mathbf{a}_{25:35}$, $_{10|}a_{25:35}$ and $_{10|}\mathbf{a}_{25:35}$.

5. Extend formula (10_1) to three lives.

98. The value at the ages x, y of any n-year temporary joint life annuity. Let $x + t$, $y + t$ denote the ages when the first rent payment is made. By definition 1, Art. 95, the value, V, at the ages x, y is given by

$$V = R\frac{v^t l_{x+t:y+t} + v^{t+1} l_{x+t+1:y+t+1} + \cdots + v^{t+n-1} l_{x+t+n-1:y+t+n-1}}{l_{xy}}$$

$$= R\frac{v^{\frac{x+y}{2}+t} l_{x+t:y+t} + v^{\frac{x+y}{2}+t+1} l_{x+t+1:y+t+1} + \cdots + v^{\frac{x+y}{2}+n-1} l_{x+t+n-1:y+t+n-1}}{v^{\frac{x+y}{2}} l_{xy}}$$

$$= R\frac{D_{x+t:y+t} + D_{x+t+1:y+t+1} + \cdots + D_{x+t+n-1:y+t+n-1}}{D_{xy}}$$

$$= R\frac{N_{x+t:y+t} - N_{x+t+n:y+t+n}}{D_{xy}} \tag{10}$$

The symbols for the values of temporary joint life annuities of annual rent 1 are analogous to those for single lives given in Art. 81. From formula (10), it follows that

$$a_{xy\overline{n}|} = \frac{N_{x+1:y+1} - N_{x+1+n:y+1+n}}{D_{xy}}, \quad \mathbf{a}_{xy\overline{n}|} = \frac{N_{x+y} - N_{x+n:y+n}}{D_{xy}}$$

$${}_{n_1|}a_{xy\overline{n}|} = \frac{N_{x+1+n_1:y+1+n_1} - N_{x+1+n_1+n:y+1+n_1+n}}{D_{xy}},$$

$${}_{n_1|}\mathbf{a}_{xy\overline{n}|} = \frac{N_{x+n_1:y+n_1} - N_{x+n_1+n:y+n_1+n}}{D_{xy}}$$

EXERCISES

1. Apply formula (10) to write the expressions for the values at the ages 35, 40, at the ages 45, 50, and at the ages 55, 60, of the 10-year temporary joint life annuity of annual rent $1000 represented by the following diagram:

Ages 35	36 · · · 45	46	47 · · · 54	55	
Ages 40	41 · · · 50	51	52 · · · 59	60	
		1000	1000 · · · 1000	1000	

2. Apply formula (10) to write the expressions for the net single premiums at the ages 25, 35 of the following joint life annuity policies each having an annual rent of $1000:

(a) Ordinary 20-year temporary joint life annuity due,

(b) Ten year temporary joint life annuity immediate deferred 5 years.

3. Use formula (10) to write the expressions for $a_{35:40\,\overline{10}|}$, $\mathbf{a}_{35:40\,\overline{10}|}$, ${}_{10|}a_{35:40\,\overline{10}|}$ and ${}_{10|}\mathbf{a}_{35:40\,\overline{10}|}$.

4. Show that formula (10) includes formulas (10_1) and (10_2).

5. Show that the value of a forborne n-year temporary joint life annuity due is given by

$$V = R\frac{N_{xy} - N_{x+n:y+n}}{D_{x+n:y+n}}$$

where x, y are the ages at the beginning of the term of the annuity. (See formula (2), Art. 82.)

99. Joint life insurance. In a single life insurance the benefit is paid when the insured dies, providing death takes place during the term of the insurance. In a joint life insurance the benefit is paid when any one of the insured dies, providing death takes place during the term of the insurance. The discussion of joint life insurances is entirely analogous to that of joint life annuities.

100. The value of joint life insurance defined. If each of the l_{28} persons living at age 28 shown in the mortality table is paired with each of the l_{32} persons living at age 32, there are $l_{28}l_{32}$ such pairs. Of these pairs $l_{29}l_{33}$ survive one year, $l_{30}l_{34}$ survive two years, and so on. It follows that $l_{28}l_{32}-l_{29}l_{33}$ pairs fail during the first year, $l_{29}l_{33}-l_{30}l_{34}$ fail during the second, and so on. If each of the $l_{28}l_{32}$ pairs of persons of ages 28, 32 holds a ten year term joint life insurance policy of face value 1, the amounts that will be received on these policies at the end of 1, 2, $\cdots$, 10 years are $l_{28}l_{32}-l_{29}l_{33}$, $l_{29}l_{33}-l_{30}l_{34}$, $\cdots$, $l_{37}l_{41}-l_{38}l_{42}$ respectively. The sum of these amounts, valued as of the ages 28, 32 at an annual interest rate i, is $v(l_{28}l_{32}-l_{29}l_{33}) + v^2(l_{29}l_{33}-l_{30}l_{34}) + \cdots + v^{10}(l_{37}l_{41}-l_{38}l_{42})$ where $v = \frac{1}{(1+i)}$. The quotient

$$V = \frac{v(l_{28}l_{32}-l_{29}l_{33})+v^2(l_{29}l_{33}-l_{30}l_{34})+ \cdots +v^{10}(l_{37}l_{41}-l_{38}l_{42})}{l_{28}l_{32}}$$

gives what is called the value at the ages 28, 32, at an interest rate i, of this term joint life insurance as determined by the mortality table. The sum of the amounts due the pairs of beneficiaries valued as of the ages 38, 42 at an interest rate i is $(1+i)^9(l_{28}l_{32}-l_{29}l_{33})+ \cdots + (l_{37}l_{41}-l_{38}l_{42})$. The quotient

$$V = \frac{(1+i)^9(l_{28}l_{32}-l_{29}l_{33})+(1+i)^8(l_{29}l_{33}-l_{30}l_{34})+ \cdots +(l_{37}l_{41}-l_{38}l_{42})}{l_{38}l_{42}}$$

gives what is called the value at the ages 38, 42, at an interest rate i, of this term joint life insurance as determined by the mortality

table. In other words, the value at ages 28, 32 of this ten year term joint life insurance is the amount that each of the $l_{28}l_{32}$ pairs of insured must pay to a company so that the total of these amounts with interest will be just enough to enable the company to pay 1 to each pair of beneficiaries at the end of each year of the term of the insurance. Likewise, the value at ages 38, 42 of this joint life insurance is the amount that each of the $l_{38}l_{42}$ pairs of persons would receive if the benefits were not drawn when they are due but were allowed to accumulate at compound interest to the time at the ages 38, 42 and if the fund thus created were then equally divided among the $l_{38}l_{42}$ survivors. The values at the pairs of ages 28, 32, and 38, 42 of this term joint life insurance as just defined are included in the

Definition 1. *The value at the ages x, y of the insured of any joint life insurance of face value, F, is the quotient obtained by dividing l_xl_y into the sum of the values obtained by applying the compound interest formula to each sum of the set consisting of F times the number of pairs that fail, according to the mortality table, during each year of the insurance term.*

From this definition the value at the ages x, y of an ordinary or of a deferred joint life insurance of face value F may be viewed as the amount that each of the l_xl_y pairs of insured must pay to a company so that the total of these amounts with interest will be just enough to enable the company to pay F to each pair of beneficiaries at the end of each year during the term of the insurance. Likewise the value at the ages x, y of a forborne joint life insurance whose term begins n or more years prior to the ages x, y, where n is the number of years in the term of the insurance, may be viewed as the amount that each of the l_xl_y pairs of survivors would receive if the benefits were not drawn when they are due but were allowed to accumulate at compound interest to the time at the ages x, y and if the fund thus created were then equally divided among the l_xl_y pairs of survivors.

When the joint life insurance in definition 1 is for a one year term beginning at the ages $x+t-1$, $y+t-1$, the definition can be stated in the form:

The value, V, at the ages x, y of a joint life insurance of face value, F, for the year beginning at the ages $x + t - 1, y + t - 1$ is given by

$$V = Fv^t \cdot \frac{l_{x+t-1}l_{y+t-1} - l_{x+t}l_{y+t}}{l_x l_y} = F\frac{C_{x+t-1:y+t-1}}{D_{xy}} \qquad (11_2)$$
$$= Fv^t \cdot {}_{t-1|}q_{xy} \qquad \text{(when } t > 0\text{)}$$

where $$C_{xy} = v^{\frac{x+y}{2}+1}(l_x l_y - l_{x+1}l_{y+1}),$$

so that $$C_{x+t-1:y+t-1} = v^{\frac{x+y}{2}+t}(l_{x+t-1}l_{y+t-1} - l_{x+t}l_{y+t}).$$

The second form of formula (11_2) is obtained from the first by multiplying numerator and denominator by $v^{\frac{x+y}{2}}$. When t is positive, V is the present or discounted value at the ages x, y of a benefit, F, to be paid at the end of t years in case two persons, aged x and y, both survive $t - 1$ years but both do not survive t years. When t is negative, V is the amount at the ages x, y of the benefit, F, paid at the ages $x + t$, $y + t$, at the end of a one year term joint life insurance, which is allowed to accumulate for t years in accordance with formula (11_2).

In finding the values at two or more pairs of ages of a joint life insurance use can be made of the following

Theorem VII. *If $V_{x+t:y+t}$ denotes the value at the ages $x + t$, $y + t$ of a joint life insurance, its value V_{xy} at the ages x, y is given by*

$$V_{xy} = v^t\frac{l_{x+t}l_{y+t}}{l_x l_y}V_{x+t:y+t} = \frac{D_{x+t:y+t}}{D_{xy}}V_{x+t:y+t}.$$

This theorem is an extension of Theorem V. The proof is left as an exercise. When t is positive, $\frac{D_{x+t:y+t}}{D_{xy}}$ is a *discount factor;* when t is negative, it is an *accumulation factor*. It should be noted that Theorem VII includes formula (11_2).

In writing the equations needed for finding the unknowns in problems involving life annuities or life insurances, use can be made of

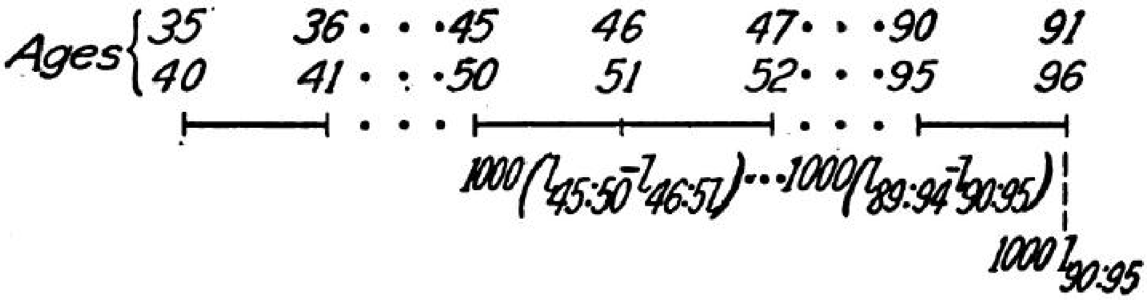
Theorem VIII. *If two joint life annuities, two joint life insurances, or a joint life annuity and a joint life insurance have equal values at the ages $x + t$, $y + t$, they have equal values at the ages x, y.*

This theorem is seen to be true by noting that Theorems VI and VII have the same discount or accumulation factor, $\dfrac{D_{x+t:y+t}}{D_{xy}}$.

The value at the ages x, y of the joint life insurance specified by any policy is called the net single premium of the policy.

EXERCISES

1. Apply definition 1 to the set of sums below the line in the following diagram to write the expressions for the values at the ages 35, 40, and at the ages 45, 50 of a joint whole life insurance of face value \$1000 whose term begins at the ages 45, 50:



2. Use definition 1 and $i = .035$ to compute the value at the ages 90, 92 of an ordinary joint whole life insurance of face value \$1000. Ans. \$958.84.

3. A one year term joint life insurance of \$1000 begins at the ages 40, 50. By use of formula (11_2) find its value at the ages 60, 70 and at the ages 20, 30. Ans. \$109.99; \$7.89.

101. Another definition of a value of a joint life insurance. The expressions given in Art. 100 for the values at the pairs of ages 28, 32 and 38, 42 of a ten year term joint life insurance whose term begins at the ages 28, 32 can be written in the forms:

$$V = v\frac{l_{28}l_{32}-l_{29}l_{33}}{l_{28}l_{32}} + v^2\frac{l_{29}l_{33}-l_{30}l_{34}}{l_{28}l_{32}} + \cdots + v^{10}\frac{l_{37}l_{41}-l_{38}l_{42}}{l_{28}l_{32}} \text{ (at ages 28, 32)}$$

$$V = v^{-9}\frac{l_{28}l_{32}-l_{29}l_{33}}{l_{38}l_{42}} + v^{-8}\frac{l_{29}l_{33}-l_{30}l_{34}}{l_{38}l_{42}} + \cdots + \frac{l_{37}l_{41}-l_{38}l_{42}}{l_{38}l_{42}} \text{ (at ages 38, 42)}$$

These forms show that the value of this insurance at either pair of ages is the sum of the values obtained by applying formula (11_2) to each sum of the set consisting of 1 payable at the end of each year of the term of the insurance. The value of the term life insurance defined in this way is included in the

Definition 2. *The value at the ages x, y of any joint life insurance of face value, F, is the sum of the values obtained by applying the formula*

$$V = Fv^t \frac{l_{x+t-1}l_{y+t-1} - l_{x+t}l_{y+t}}{l_x l_y} = F\frac{C_{x+t-1:y+t-1}}{D_{xy}}$$

to each sum of the set consisting of F payable at the end of each year of the term of the insurance.

From this definition and the value of an expectation (Art. 76), the value at the ages x, y of an ordinary or of a deferred joint life insurance of face value, F, is the sum of the values at the ages x, y of the insured, of the expectations of a pair of beneficiaries of receiving F, at the end of each year during the term of the insurance; or, it is the sum of the discounted values at the ages x, y obtained by use of formula (11_2) when t is positive, of the set of sums consisting of F at the end of each year of the term of the insurance. Likewise, the value at the ages x, y of a forborne joint life insurance whose term begins n or more years prior to the time at the ages x, y, where n is the number of years in the term of the insurance, is the sum of the amounts at the ages x, y of the same set of sums when each is accumulated by use of formula (11_2) when t is zero or negative.

EXERCISE

Apply definition 2 to the set of sums below the line in the following diagram to write the expressions for the values at the ages 35, 40, at the ages 45, 50, and at the ages 55, 60, of a 10-year term joint life insurance of face value \$1000 whose term begins at the ages 45, 50.

Ages	35	36 · · · 45	46	47 · · · 54	55
	40	41 · · · 50	51	52 · · · 59	60
			1000	1000 · · · 1000	1000

102. The value at the ages x, y of any joint whole life insurance of face value F. Let $x + k$, $y + k$ denote the ages when the term of the insurance begins. By definition 1, Art. 100, the value, V, at the ages x, y is given by

$$V = F\,\frac{v^{k+1}(l_{x+k:y+k} - l_{x+k+1:y+k+1}) + v^{k+2}(l_{x+k+1:y+k+1} - l_{x+k+2:y+k+2}) + \text{etc.}}{l_{xy}}$$

$$= F\,\frac{v^{\frac{x+y}{2}+k+1}(l_{x+k:y+k} - l_{x+k+1:y+k+1}) + v^{\frac{x+y}{2}+k+2}(l_{x+k+1:y+k+1} - l_{x+k+2:y+k+2}) + \text{etc.}}{v^{\frac{x+y}{2}}\,l_{xy}}$$

$$= F\,\frac{C_{x+k:y+k} + C_{x+k+1:y+k+1} + \text{etc.}}{D_{xy}}$$

$$= F\,\frac{M_{x+k:y+k}}{D_{xy}} \qquad (11_1)$$

where the commutation symbol M_{xy} is defined by

$$M_{xy} = C_{xy} + C_{x+1:y+1} + \text{etc.},$$

so that $$M_{x+k:y+k} = C_{x+k:y+k} + C_{x+k+1:y+k+1} + \text{etc.}$$

The symbols for the values of joint whole life insurances of face value 1 are analogous to those for single lives given in Art. 88. From formula (11_1) it follows that

$$A_{xy} = \frac{M_{xy}}{D_{xy}}, \qquad {}_{n_1|}A_{xy} = \frac{M_{x+n_1:y+n_1}}{D_{xy}}$$

EXERCISES

1. Apply formula (11_1) to write the expressions for the values at the ages 35, 40, and at the ages 45, 50, of the joint whole life insurance of face value \$1000 represented by the following diagram:

Ages	35	36 · · · 45	46	47 · · · 90	91
	40	41 · · · 50	51	52 · · · 95	96
			1000	1000 · · · 1000	1000

2. Find the expression for the value at the ages 35, 40, of the joint whole life insurance in Exercise 1 by applying Theorem VII to its value at the ages 45, 50.

3. Apply formula (11_1) to write the expressions for the net single premiums at the ages 25, 35, of the following joint life policies each having a face value of \$1000:

(*a*) Ordinary joint whole life insurance,

(*b*) Joint whole life insurance deferred 10 years.

4. Use formula (11_1) to write the expressions for

$$A_{25:35} \quad \text{and} \quad {}_{10|}A_{25:35}.$$

5. Extend formula (11_2) to three lives.

103. The value at the ages x, y of any joint n-year term insurance of face value F. Let $x+k$, $y+k$ denote the ages when the term of the insurance begins. By definition 1, Art. 100, the value, V, at the ages x, y is given by

$$V = F\,\frac{v^{k+1}(l_{x+k:y+k} - l_{x+k+1:y+k+1}) + \cdots + v^{k+n}(l_{x+k+n-1:y+k+n-1} - l_{x+k+n:y+k+n})}{l_{xy}}$$

$$= F\,\frac{v^{\frac{x+y}{2}+k+1}(l_{x+k:y+k} - l_{x+k+1:y+k+1}) + \cdots + v^{\frac{x+y}{2}+k+n}(l_{x+k+n-1:y+k+n-1} - l_{x+k+n:y+k+n})}{v^{\frac{x+y}{2}}\, l_{xy}}$$

$$= F\,\frac{C_{x+k:y+k} + \cdots + C_{x+k+n-1:y+k+n-1}}{D_{xy}}$$

$$= F\,\frac{M_{x+k:y+k} - M_{x+k+n:y+k+n}}{D_{xy}} \tag{11}$$

The symbols for the values of joint life term insurances of face value 1 are analogous to those for single lives given in Art. 89. From formula (11) it follows that

$$A_{xy\overline{n|}} = \frac{M_{xy} - M_{x+n:y+n}}{D_{xy}}, \qquad {}_{n_1|}A_{xy\overline{n|}} = \frac{M_{x+n_1:y+n_1} - M_{x+n_1+n:y+n_1+n}}{D_{xy}}$$

EXERCISES

1. Apply formula (11) to write the expressions for the values at the ages 35, 40, at the ages 45, 50, and at the ages 55, 60, of the 10 year term joint life insurance of face value $1000, represented by the following diagram:

Ages	35	36 ... 45	46	47 ... 54	55
	40	41 ... 50	51	52 ... 59	60
			1000	1000 ... 1000	1000

2. Apply formula (11) to write the expressions for the net single premiums at the ages 25, 35, of the following joint life insurance policies each having a face value of $1000:

(*a*) Ordinary 20 year term life insurance.

(*b*) Ten year term joint life insurance deferred 5 years.

3. Use formula (11) to write the expressions for

$$A_{35:40\,\overline{10|}} \text{ and } {}_{10|}A_{35:40\,\overline{10|}}$$

4. Show that formula (11) includes formula (11_1) and formula (11_2).

5. Show that the value of a forborne n-year term joint life insurance is given by

$$V = F\frac{M_{xy} - M_{x+n:y+n}}{D_{x+n:y+n}}$$

where x, y are the ages at the beginning of the term of the insurance. (See formula 6, Art. 90.)

6. Use formulas (11) and (10_3) to write the expressions for the value at the ages 25, 30 of a 20-year joint life endowment insurance.

104. Computation of the values of joint life annuities and of joint life insurances. A table giving the values at a given interest rate of any one of the commutation symbols D_x, N_x, C_x, M_x, R_x, S_x for each of the 86 ages in the American Experience Table requires only 86 entries. Tables giving the values of any one of the corresponding symbols for two, three, four, or more lives would require large numbers of entries. Such tables would be impractical to prepare. There are two practical methods in common use for computing the value of any joint life annuity or insurance. One of these is based on replacing lives of unequal ages by an equal number of lives of equivalent equal ages; the other is based on replacing lives of unequal ages by a single life in which case an appropriate change in the interest rate is needed.* An important property of the method of equal ages may be stated in the form (using two ages): *If w, w are the equal ages which are equivalent to the ages x, y then $w + t$, $w + t$ are the equal ages which are equivalent to the ages $x + t$, $y + t$.* This property is called the *Law of Uniform Seniority*. From this property it follows that formulas (10) and (11) may be written in the forms:

$$V = R\frac{N_{w+t:w+t} - N_{w+t+n:w+t+n}}{D_{ww}} \qquad (10')$$

$$V = F\frac{M_{w+k:w+k} - M_{w+k+n:w+k+n}}{D_{ww}} \qquad (11')$$

* For brief expositions of these two methods reference may be made to a paper by H. L. Rietz in Volume XXVIII of the American Mathematical Monthly, page 158 (1921), or to Mathematical Theory of Life Insurance by C. H. Forsyth — John Wiley and Son (1924).

In this article some examples are solved to show how values of joint life annuities and insurances can be computed by the use of tables giving the values of joint life commutations symbols for equal ages and of μ_x, the "force of mortality" at the age x. By μ_x is meant the instantaneous yearly death rate at the age x. In the yearly death rate, $q_x = \frac{d_x}{l_x}$, d_x is the actual number of deaths among the l_x persons during the year following the age x; in the force of mortality the denominator is l_x but the numerator is the number of deaths that would take place among the l_x persons during the year following the age x, if the rate throughout the year remained the same as the instantaneous rate at age x.

In Table XIV are given Hunter's Makehamized American Experience Mortality Table, values of the force of mortality, and values at $3\frac{1}{2}\%$ of the commutation symbols for two equal ages. In the table below are given for certain ages the values at 5% of the commutation symbols for a single life according to the American Experience Table and for two lives of equal ages according to Hunter's Makehamized American Experience Table:

Age x	D_x	N_x	M_x	μ_x	D_{xx}	N_{xx}	M_{xx}
25	26291.40	435657.09	5545.82	.00804	23396.19	339371.57	72356.38
26	24837.49	409365.69	5343 89	.00809	21925.55	315975.38	68791.01
27	23462.44	384528.20	5151.57	.00814	20545.12	294049.84	65427.49
28	22162.03	361065.76	4968.42	.00821	19249.49	273504.71	62254.58
29	20932.25	338903.73	4793.98	.00827	18033.56	254255.22	59261.65
30	19769.12	317971.48	4627.62	.00835	16891.33	236221.66	56426.79
31	18669.07	298202.35	4468.93	.00843	15819.24	219330.33	53749.39
32	17628.76	279533.28	4317.65	.00853	14812.74	203511.09	51217.36
33	16644.79	261904.52	4173.14	.00863	13867.27	188698.35	48816.35
34	15713.98	245259.74	4034.94	.00875	12979.59	174831.08	46543.01
35	14833.53	229545.76	3902.78	.00888	12145.40	161851.49	44381.88
36	14000.79	214712.22	3776.40	.00902	11361.92	149706.09	42330.61
.	.	.	.	.	.	.	.
47	7318.515	97755.471	2663.49	.01215	5297.803	58734.708	25009.12
48	6886.371	90436.956	2579.85	.01265	4922.039	53436.905	23774.25
49	6476.407	83550.585	2497.81	.01321	4567.884	48514.866	22576.52
50	6087.169	77074.178	2416.97	.01384	4234.369	43946.982	21416.56
51	5717.409	70987.009	2337.07	.01453	3919.986	39712.613	20289.09
52	5365.975	65269.600	2257.90	.01531	3623.609	35792.627	19191.98
53	5031.809	59903.625	2179.25	.01617	3344.111	32169.018	18122.53
54	4713.927	54871.816	2100.98	.01712	3080.668	28824.907	17080.54

The value of w for a given set of n ages x, y, $\cdots$ is found as follows:

(1) Find μ_w by means of the relation $\mu_w = \dfrac{\mu_x + \mu_y + \text{etc.}}{n}$;

(2) Find w by simple interpolation in the table for x and μ_x.

Example 1. Find the equal ages equivalent to the ages 25, 27, 34.

Solution. In this example, $\mu_w = \dfrac{\mu_{25} + \mu_{27} + \mu_{34}}{3} = .00831$. Simple interpolation in the table for x and μ_x shows that $w = 29.75$.

Example 2. Find the equal ages equivalent to the ages 28, 32.

Solution. In this example $\mu_w = \dfrac{\mu_{28} + \mu_{32}}{2} = .00837$. Simple interpolation in the table for x and μ_x shows that $w = 30.25$.

Example 3. Find the value of $a_{28:32}$ at 5%.

Solution. By formula (10′)

$$
\begin{aligned}
a_{28:32} = a_{ww} &= \frac{N_{w+1:w+1}}{D_{ww}} \text{ where, by Example 2, } w = 30.25 \\
&= \frac{215375.52}{16623.31} \text{ (by interpolation)} \\
&= 12.956
\end{aligned}
$$

Example 4. Find the value of $A_{28:32}$ at $3\frac{1}{2}$%.

Solution. By formula (11′)

$$
\begin{aligned}
A_{28:32} = A_{ww} &= \frac{M_{ww}}{D_{ww}} \text{ where, by Example 2, } w = 30.25 \\
&= \frac{11339.33}{25685.03} \\
&= .44148
\end{aligned}
$$

EXERCISES

1. Solve Example 3 at $3\frac{1}{2}$%.
2. Solve Example 4 at 5%. Ans. .33542.
3. Find the value at 5% of $a_{30:50}$.
4. Find the value at 5% of $A_{25:\overline{10|}}$.
5. Find the value at 5% of $a_{25:\overline{10|}}$.
6. Find the value at 5% of $A_{30:53}$.

105. Survivorship annuities and insurances. The annuities and insurances considered in Arts. 94 to 104 inclusive are based on the joint lives of two or more persons. Combinations of these can also be made which are analogous to those considered in Arts. 91 and 93 for single life annuities and insurances. Other annuities and insurances of importance are based on one or more *survivors* of a set of joint lives. A brief treatment is given in this article of a few simple annuities and insurances which are based on the survivor of two persons (x) and (y) of ages x and y respectively. The values of these annuities and insurances, as well as of many others, can be written at once by resolving them into annuities and insurances based on a single life, or on joint lives.

The value, V, of a life annuity immediate of annual rent, R, payable during the joint lives of (x) and (y) and during the life of the survivor is given by

$$V = R(a_x + a_y - a_{xy})$$

Such an annuity is called an annuity for the life of the last survivor. When $R = 1$, V is denoted by $a_{\overline{xy}}$ so that

$$a_{\overline{xy}} = a_x + a_y - a_{xy}.$$

The value, V, of a life annuity immediate of annual rent, R, payable during the life of a designated one [(x) or (y)] after the death of the other [(y) or (x)] is given by

$$V = R(a_x - a_{xy}) \text{ if } (x) \text{ is the survivor, and by}$$
$$V = R(a_y - a_{xy}) \text{ if } (y) \text{ is the survivor.}$$

In the first case the annuity is called a survivorship or reversionary annuity on the life of (x) beginning at the death of (y). When $R = 1$, V is denoted by $a_{y|x}$, so that

$$a_{y|x} = a_x - a_{xy}.$$

The value, V, of a life annuity immediate of annual rent, R, payable during the life of either one but after the death of the other is given by

$$V = R(a_x + a_y - 2\,a_{xy})$$

This is a combination of the two cases in the preceding survivorship annuity.

The value, V, of a joint life annuity immediate of annual rent, $\frac{R}{2}$, combined with a survivorship annuity of annual rent, R, is given by $V = \frac{R}{2}\, a_{xy} + R(a_x - a_{xy})$, if x is the survivor; that is,

$$V = R(a_x - \tfrac{1}{2}\, a_{xy}) \text{ if } (x) \text{ is the survivor.}$$

Similarly $V = R(a_y - \frac{1}{2}\, a_{xy})$ if (y) is the survivor.

The value, V, of a life insurance of face value, F, payable upon the death of the survivor of (x) and (y) is given by

$$V = F(A_x + A_y - A_{xy}).$$

Corresponding formulas can be written for more than two lives.

EXERCISES

1. Compute the values at $3\frac{1}{2}\%$ and at 5% of $a_{30:53}$.
2. Compute the values at $3\frac{1}{2}\%$ and at 5% of $a_{30\mid 53}$.
3. Interpret each of the following:

(a) $\mathbf{a}_{x|y} = \mathbf{a}_y - \mathbf{a}_{xy}$,

(b) $\mathbf{a}_{\overline{xy}} = \mathbf{a}_x + \mathbf{a}_y - \mathbf{a}_{xy}$,

(c) $a_{\overline{xyz}} = a_x + a_y + a_z - a_{yz} - a_{xz} - a_{xy} + a_{xyz}$,

(d) $A_{\overline{xyz}} = A_x + A_y + A_z - A_{yz} - A_{xz} - A_{xy} + A_{xyz}$.

4. Show that $\mathbf{a}_{x|y} = a_{x|y}$.

CHAPTER V

APPLICATION OF LIFE ANNUITIES AND OF LIFE INSURANCES

106. Introduction. In Chapter IV formulas are derived for finding the values at age x of life annuities and of life insurances and a few simple applications are given. In this chapter the applications are treated more fully and systematically. The suggestions given in Art. 33, Chapter III, for solving problems based on annuities certain are helpful for solving problems based on life annuities and life insurances.

The applications are arranged under three headings as follows: first, Premiums; second, Valuation of Policies, Cost of Insurance, Dividends, Policy Options; third, Life Estates, Remainders, Inheritance Taxes.

Premiums

107. Premiums defined. As stated in Arts. 78, 86, and 91, Chapter IV, the value of a life annuity, of a life insurance, or of a combined life annuity and life insurance which is specified by a policy, is called the *net single premium* of the policy. The net single premium is the net cost of the policy at the time it is purchased. A purchaser of a policy usually finds it more convenient to make periodic payments, during the whole or a part of his life, which are equivalent to the net cost, than to pay this cost in a lump sum. Such payments are called net premiums, and they are usually made at the beginning of the payment periods. In other words the *net premiums* of a policy are the rent payments of a whole or temporary life annuity due whose value at the time the policy is purchased equals that of the net single premium of the policy at this time. These premiums are called *net level premiums* when they are equal in value. Other forms of net premiums are defined in Arts. 112, 120, 121, and 122. Net premiums provide companies with funds for the payment of policy benefits. In

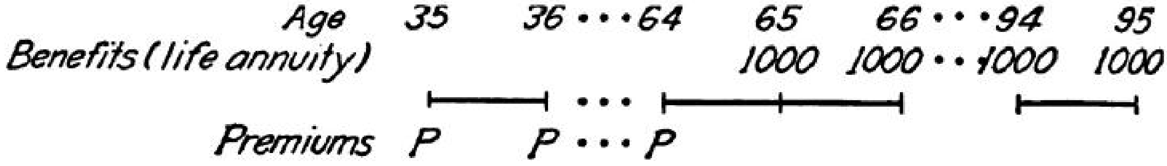
fact they usually provide more than is necessary for this purpose. This is due largely to two factors: first, companies are usually able to invest premiums at a higher rate than that used in computing them, and, second, the mortality shown by the table used in computing premiums is larger in the early years of policies than the actual mortality. In addition to funds for the payment of policy benefits companies must also have funds for the payment of interest on capital invested, for rent, taxes, salaries, and wages, agent's commissions, medical fees, and other expenses connected with the administration of its affairs. Net premiums are not sufficient to provide these additional funds. They are obtained by the addition of certain amounts, called *loadings*, to the net premiums. The net premiums plus the loadings are called the *gross* or *office* premiums. The gross premiums are those actually paid by the policyholders. The gross premiums must be ample to enable companies to pay policy benefits and expenses, as well as to build up and maintain an adequate surplus. Such a surplus is necessary since companies cannot conduct business on an exact theoretical basis. Gross premiums are considered further in Arts. 113 and 114.

The definition of a net level premium of any policy leads at once to an equation which determines its value. One member of this equation is the value at a known age of the annuity, corresponding to the form of payment of the net level premiums; the other member is the value at this age of the net single premium of the policy. Otherwise stated, one member of this equation is the value at a given age of the policy benefits and the other member is the value at this age of the premium payments. Formulas (1) and (5), Chapter IV, give these values for single lives and formulas (10) and (11) for two lives. Theorems V and VIII may be used in writing these equations. In Arts. 108 to 111 inclusive these premiums are determined for some important types of policies.

108. Net level premiums of life annuity policies. Both members of the equation which determine the net level annual premiums of life annuity policies on single lives can be written by the use of formula (1), Chapter IV. When two or more lives are involved, formula (10), Chapter IV, is needed.

EXAMPLE 1. Find the net level annual premium, P, of a \$1000 whole life annuity policy purchased at age 35 if the first benefit of the policy is payable at age 65 and if the premiums are payable annually, as long as the annuitant survives, during the term which begins at age 35 and ends at age 65.

SOLUTION. In the following diagram the benefits are above the line and the premiums below the line:



Equating the value of the premiums at age 35 to the value of the benefits at this age gives

$$P\,\frac{N_{35} - N_{65}}{D_{35}} = 1000\,\frac{N_{65}}{D_{35}}$$

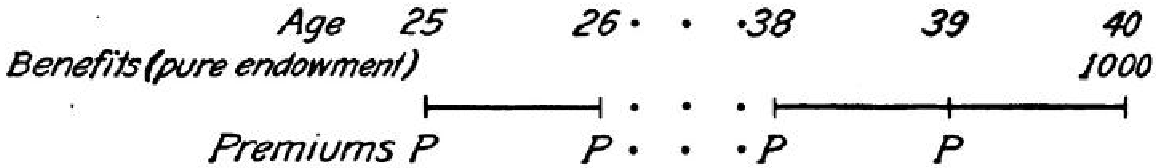
Solving,
$$P = 1000\,\frac{N_{65}}{N_{35} - N_{65}} = \$119.08$$

EXERCISE 1. Solve Example 1 by equating values at age 65.

EXERCISE 2. Solve Example 1 if the premiums are payable annually from age 35 to age 50.

EXAMPLE 2. Find the net level annual premium, P, of a \$1000 pure endowment policy purchased at age 25 and due in 15 years if the premiums are payable annually, as long as the annuitant survives, during the term which begins at age 25 and ends at age 40.

SOLUTION. In the following diagram the benefits are above the line and the premiums below the line:



Equating the value of the premiums at age 25 to the value of the benefit at this age gives,

$$P\,\frac{N_{25} - N_{40}}{D_{25}} = 1000\,\frac{D_{40}}{D_{25}}$$

Solving,
$$P = 1000\,\frac{D_{40}}{N_{25} - N_{40}} = \$42.40$$

EXERCISE 1. Solve Example 2 by equating values at age 40.

EXERCISE 2. Solve Example 2 if the premiums are paid annually from age 25 to age 35.

EXAMPLE 3. Find the net level annual premium, P, of a \$1000 annuity policy purchased at age 35 whose first \$1000 is paid at age 65 if the policy provides a ten-year annuity certain followed by a whole life annuity, and if the premiums are payable annually from age 35 to age 65. Use $i = .035$.

SOLUTION. In this case the policy benefits consist of a ten-year annuity certain, followed by a whole life annuity. Equating the value of the premiums at age 65 to that of the benefits at this age gives

$$P\frac{N_{35} - N_{65}}{D_{65}} = 1000\,(1 + a_{\overline{9}|.035}) + 1000\frac{N_{75}}{D_{65}}$$

Solving, $$P = \$651.22$$

EXERCISES

1. Compute the net level annual premium, P, payable as a life annuity due from age 25 to age 50 of a pure endowment policy of \$1000 due at age 50.

2. Same as Exercise 1 except the \$1000 is due at age 60.

3. Same as Exercise 1 for a whole life annuity policy of \$1000 annual rent with first payment at age 50.

4. Same as Exercise 1 except that the premiums are payable quarterly instead of annually. Use formula (4), Art. 84.

5. Same as Exercise 1 for a 20-year temporary life annuity policy of \$1000 annual rent with first payment at age 50.

6. Compute the net level annual premium, P, payable as a joint life annuity due from the ages 25, 30 to the ages 50, 55 of a joint life pure endowment policy of \$1000 due at the ages 50, 55.

7. Same as Exercise 6 for a joint whole life annuity policy of \$1000 annual rent with first payment at the ages 26, 31.

8. Same as Example 6 with first payment at ages 50, 55.

109. Net level premium formulas for life annuity policies. A general formula for the net level premium at age x, corresponding to a prescribed form of payment, can readily be found for any life annuity policy by the method of equating the value of the premiums at age x to that of the benefits at this age. For an n-payment, n-year pure endowment policy of rent, R, issued at age x, whose premiums, P, are payable annually as a life annuity due from age x to age $x + n$, the value of P is given by

$$P = R\frac{D_{x+n}}{N_x - N_{x+n}}$$

In this case the equation of value at age x is $P\dfrac{N_x - N_{x+n}}{D_x} = R\dfrac{D_{x+n}}{D_x}$;

the value of P given above is found by solving this equation for P.

When $R = 1$, P is denoted by $\frac{1}{{}_n u_x}$ (Art. 82, Chap. IV), so that for an *n-payment, n-year pure endowment policy of face value* 1

$$P = \frac{1}{{}_n u_x} = \frac{D_{x+n}}{N_x - N_{x+n}}$$

Similarly for an *n-payment whole life annuity of annual rent, R,* issued at age x with first payment at age $x + n$, whose premiums, P, are payable annually as a life annuity due from age x to age $x + n$, the value of P is given by

$$P = R \frac{N_{x+n}}{N_x - N_{x+n}}$$

Since these and like formulas for the net level premiums of other life annuity and of life insurance policies can be derived so easily, the student is advised to write the equation which determines the value of P in each case rather than to substitute into a formula.

EXERCISE

Derive the formula for the net level annual premium, P, at age x of an n-year temporary life annuity policy of annual rent, R, with first payment at age $x + t$ if the premiums are payable as a life annuity due from age x to age $x + t$; from age x to age $x + m$.

110. Net level premiums of life insurance policies. Notation. The benefits of these policies consist of life insurances so that the formulas for the values of life insurances and of annuities are needed to write the equations which determine the net level premiums of these policies.

Example 1. Find the net level annual premium, P, of a 20-payment whole life insurance policy of \$1000 issued at age 30.

Solution. In the following diagram the benefits are above the line and the premiums below the line:

Age	30	31 · · · 49	50 · · · 95	96
Benefits (life insurance)		1000 · · · 1000	1000 · · · 1000	1000
Premiums	P	P · · · P		

Equating the value of the premiums at age 30 to the value of the benefit at this age gives

$$P\,\frac{N_{30} - N_{50}}{D_{30}} = 1000\,\frac{M_{30}}{D_{30}}$$

Solving,

$$P = 1000\,\frac{M_{30}}{N_{30} - N_{50}} = \$24.71$$

Example 2. Find the net level annual premium, P, of an ordinary 20-year endowment insurance policy of \$1000 issued at age 30.

Solution. In the following diagram the benefits are above the line and the premiums below the line:

		30	31	...	49	50
Benefits	*pure endowment*					*1000*
	term insurance		*1000*	...	*1000*	*1000*
Premiums		*P*	*P*	...	*P*	

Equating the value of the premiums at age 30 to the value of the benefits at this age gives

$$P\,\frac{N_{30} - N_{50}}{D_{30}} = 1000\,\frac{M_{30} - M_{50} + D_{50}}{D_{30}}$$

Solving,

$$P = 1000\,\frac{M_{30} - M_{50} + D_{50}}{N_{30} - N_{50}} = \$39.51$$

Example 3. Same as Example 2 except that the premiums are payable quarterly.

Solution. Equating values at age 30, using formula, (4) Chapter IV, and solving for P, the quarterly premium, gives

$$P = \frac{1000}{4}\,\frac{M_{30} - M_{50} + D_{50}}{N_{30} - N_{50} - \frac{3}{8}(D_{30} - D_{50})} = \$10.04$$

Example 4. Find the net level annual premium, P, of a 20-payment increasing insurance of \$1000 issued at age 25 for a term of 20 years.

Solution. Equating values at age 25, using formula (8), Chapter IV, and solving for P, gives

$$P = 1000\,\frac{R_{25} - R_{45} - 20\,M_{45}}{N_{25} - N_{45}} = \$82.08$$

Example 5. Find the net level annual premium, P, of a \$1000 ordinary joint whole life policy issued at the ages 28, 32.

Solution. Equating values at the ages 28, 32 and solving for P gives

$$P = 1000\,\frac{M_{28:32}}{N_{28:32}} = \$26.73$$

EXERCISES

1. Compute the net level annual premium of a $1000 ordinary whole life insurance policy issued at age 35. Ans. $19.91.

2. Same as Exercise 1 for a 20-payment whole life insurance policy. Ans. $27.40.

3. Same as Exercise 1 for a 20-year endowment insurance policy. Ans. $40.12.

4. Same as Exercise 1 for a 10-payment, 20-year endowment insurance policy.

5. Same as Exercise 1 for a 10-payment, 10-year term insurance policy.

6. Same as Exercise 1 for a 1-payment, 1-year term insurance policy. Ans. $8.64.

111. Net level premium formulas for life insurance policies. Notation. A general formula for the net level premium at age x corresponding to a prescribed form of payment, can readily be found for any life insurance policy by the method of equating the value of the premiums at age x to the value of the benefits at this age. There is a standard notation for the net level premiums of the various forms of life insurance policies of face value 1 issued at age x. In what follows in this article this notation is given for the common forms of policies, and the formulas for the values of the premiums are written. In the ordinary whole life insurance policy the premiums are payable annually as a whole life annuity due beginning at age x; in the *limited payment* policies the premiums are payable annually as a temporary life annuity due beginning at age x and continuing during a term which is equal to the number of payments. In each of the following policies, the age of issue is x and the face value is 1.

Ordinary whole life insurance policy

$$P_x = \frac{M_x}{N_x};$$

m-payment whole life insurance policy

$$_mP_x = \frac{M_x}{N_x - N_{x+m}};$$

n-payment, n-year term insurance policy

$$P^{1}_{x\overline{n}|} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}};$$

m-payment, n-year endowment insurance policy

$$ {}_mP_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+m}}; $$

n-payment, n-year endowment insurance policy

$$ P_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}} $$

The last policy is usually called an n-year endowment insurance policy.

Exercise. Verify each of the above formulas by equating the value of the premiums at age x to that of the benefits at this age and solving for the premium.

The above notation is extended to premiums of joint life insurance policies by replacing x by xy, and to premiums payable m times a year by affixing (m) in parenthesis to the upper right of the symbols. For example, P_{xy} denotes the net level premium payable annually of a joint whole life insurance policy of face value 1 for two lives, and $P_x^{(m)}$ denotes the net level premium payable m times a year of a whole life insurance policy of face value 1 for a single life.

EXERCISES

1. Show that $A_x = P_x(1 + a_x) = P_x\mathbf{a}_x$. Interpret this relation verbally.

2. Use the relations

$$ A_x = 1 - d\mathbf{a}_x \qquad \text{(Art. 92)} $$
$$ A_x = P_x\mathbf{a}_x \qquad \text{(Exercise 1)} $$

to show that

(a) $P_x = \dfrac{1}{\mathbf{a}_x} - d$

(b) $P_x = \dfrac{A_x}{1 - dA_x}$

(c) $A_x = \dfrac{P_x}{P_x + d}$

(d) $\mathbf{a}_x = \dfrac{1 - A_x}{d}$

(e) $\mathbf{a}_x = \dfrac{1}{P_x + d}$

Interpret each of these formulas verbally.

3. Write the formulas for $P_{25:30}$, and $P_{25:30\,\overline{20}|}$.

4. Write the formulas for $P_{25}^{(4)}$, ${}_{20}P_{25}^{(4)}$, and $P_{25\,\overline{20}|}^{(4)}$.

112. Natural Premiums. Full preliminary term net premiums. The net single premium for a one-year term insurance issued at age x is called the *natural premium* at age x of this insurance. By putting $F = t = 1$ in formula (5_2), Chapter IV, it is seen that the natural premium, $A^{1}_{x\,\overline{1|}}$,* of an insurance of 1 issued at age x is given by

$$A^{1}_{x\,\overline{1|}} = \frac{C_x}{D_x}$$

The natural premiums for insurance issued one year at a time through a term of years evidently increase with the age and so differ from the net level premiums considered in Arts. 110 and 111.

The *full preliminary term net premiums* of any given life insurance policy issued at age x are defined as follows: The net premium for the first year is the natural premium at age x for the face of the policy; the net premium for each subsequent year of the premium payment period is the net level premium at age $x + 1$ of a like policy issued at age $x + 1$ for a term which ends at the same time as that of the given policy. The premiums subsequent to the first are called *renewal* premiums. For example, under the full preliminary term plan the first net premium of any life insurance policy of face value 1 issued at age 30 is denoted by $A^{1}_{30:\overline{1|}}$; when the policy is ordinary whole life each renewal premium is denoted by P_{31}; when it is a 20-payment life, each renewal premium is denoted by ${}_{19}P_{31}$; when it is a 20-year endowment, each renewal is denoted by $P_{31\,\overline{19|}}$.

Gross premiums of policies are usually level and so it follows that the full preliminary term premiums allow a larger loading for the first year than for the later years. This is usually desirable, particularly for new companies, since the expenses on a policy for the first year are heavier than for the later years. For limited payment policies, however, having large gross premiums, as for example 10-year endowment, or 10-payment whole life insurance policies, the full preliminary term net premiums provide loadings for the first year which are larger than are needed to cover the extra expenses for the year. To avoid this situation various states have

* $P^{1}_{x\,\overline{1|}}$ is also used to represent the natural premiums.

adopted laws modifying the full preliminary term net premiums. Some of these modified forms of preliminary term premiums are discussed in Arts. 120, 121, and 122 after the subject of reserves has been considered.

EXAMPLE 1. Compute the natural premium at age 30 for a \$1000 policy.

SOLUTION. The natural premium in this case is $1000\frac{C_{30}}{D_{30}} = \$8.14$.

EXAMPLE 2. Compute the value of each renewal full preliminary term net premium of a \$1000 ordinary whole life insurance policy issued at age 30.

SOLUTION. Equating the value of the renewal premiums at age 31 to the value of the insurance at this age gives

$$P_{31}\frac{N_{31}}{D_{31}} = 1000\frac{M_{31}}{D_{31}}$$

Solving,

$$P_{31} = 1000\frac{M_{31}}{N_{31}} = \$17.68$$

EXERCISES

1. Write the formulas for the full preliminary term premiums of the following life insurance policies issued at age 25:
   (a) Ordinary whole life, (c) 10-year endowment,
   (b) 30-payment life, (d) 10-payment 20-year endowment.

2. Compute the values of the premiums in Exercise 1 (b) and 1 (d).

113. Gross premiums based on net level and full preliminary net term premiums. As stated in Art. 107 gross premiums are formed by adding loadings to the net premiums, and, when invested, they must provide a company with funds for the payment of policy benefits, expenses, and surplus. A portion of the surplus provided is held for unforeseen contingencies, and the remainder is given to those to whom it belongs. In stock companies issuing only non-participating policies it goes to the stockholders. In strictly mutual companies, the insured or their beneficiaries participate in this surplus; it is paid to the insured in the form of dividends, or to the beneficiaries in a lump sum or some other form of equivalent benefit. In mixed companies, part goes to the stockholders and part to the policyholders. The loadings in the gross premiums of participating policies are usually larger than of non-participating since any excess is returned to the insured. In this article no attempt is made to discuss the general theory

underlying methods of loading. A few examples will be given, however, to illustrate some of these methods for participating policies. The study of actual loading formulas affords a good way of learning the principles used. An attempt by the student to discover these formulas for policies issued by the various companies is an excellent exercise.

The following table of net premiums on $1000 policies may be used in applying loading formulas:

		Whole Life		20-Payment Life		20-Year Endowment	
Natural Premium	Age	Net Level	Preliminary Term Renewals	Net Level	Preliminary Term Renewals	Net Level	Preliminary Term Renewals
7.54	20	13.48	13.77	20.72	21.76	38.90	41.36
7.79	25	15.10	15.48	22.53	23.68	39.14	41.61
8.14	30	17.19	17.68	24.71	26.02	39.51	41.99
8.64	35	19.91	20.55	27.40	28.89	40.12	42.63
9.46	40	23.50	24.36	30.75	32.47	41.18	43.75
10.79	45	28.35	29.51	35.07	37.09	43.08	45.77
13.31	50	34.99	36.59	40.82	43.22	46.46	49.35
17.94	55	44.13	46.34	48.70	51.60	52.21	55.43
25.79	60	56.83	59.92	59.85	63.44	61.65	65.43

In the formulas given in the exercises which follow G denotes the gross premium on a $1000 policy. In some of these formulas the loadings are based on net premiums; in others, they are based on gross premiums.

EXERCISES

1. Compute the gross premiums for the ages 25 and 50 of a $1000 ordinary whole life policy by use of the formulas:

$$(a)\quad G = 1000\, P_x\, (1 + .3)$$
$$(b)\quad G = 1000\, P_x\, (1 + .2)$$
$$(c)\quad G = 1000\, P_x\, (1 + .2) + 1$$

2. Compute the gross premiums for the ages 25 and 50 of a $1000 20-payment whole life policy by use of the formulas

$$
\begin{aligned}
&(a)\ G = 1000\,[{}_{20}P_x(1 + .2) + P_x(.1)],\\
&(b)\ G = 1000\,[{}_{20}P_x(1 + .15) + P_x(.15)],\\
&(c)\ G = 1000\,[{}_{20}P_x(1 + .1) + P_x(.1)] + 1,\\
&(d)\ G = 1000\,[{}_{20}P_x(1 + .105) + P_x(.105)].
\end{aligned}
$$

3. Compute the gross premiums for the ages 25 and 50 of a $1000 20-year endowment insurance policy by use of the formulas obtained by replacing ${}_{20}P_x$ in each of the formulas given in Exercise 2 by $P_{x\overline{20|}}$.

4. Compute the gross premiums for the ages 25 and 50 of a $1000 whole life policy by use of the formula, obtained by solving $G = 1000\,P_x + .14\,G + 1$ for G,

$$G = 1000\,\frac{P_x + 1}{.86}$$

5. Compute the gross premiums for the ages 25 and 50 of a $1000 10-payment whole life policy by use of the formula

$$G = 1000\,\frac{{}_{10}P_x + 1}{.86}$$

6. Same as Exercises 1, 2, 3, 4, and 5 when the full preliminary term renewal premiums are used in place of the net level premiums.

7. Determine the amount of loading in each gross premium in Exercises 1 to 5 inclusive.

8. Determine the amount of loading for the first year and for each subsequent year in each gross premium in Exercise 6.

9. Same as Exercises 1 and 4 when P_x is replaced by $P_x^{(2)}$.

10. Compute the gross premiums payable semi-annually for the ages 25 and 50 of a $1000 whole life policy by multiplying the results found in Exercise 1 by .52. (Note that this method is simpler than that of Exercise 9.)

11. Interpret each of the formulas in Exercise 2 verbally.

114. Gross premiums on return premium policies. The benefits provided by some policies include the return of the gross premiums without interest. These gross premiums constitute an increasing insurance. In this article the gross premiums are found for a few policies which provide for the return of the gross premiums.

Example 1. Find the gross premium for an increasing insurance of $1000 issued at age 35 for a term of 30 years if the loading is 13 per cent of the gross premium.

Solution. By the method in Example 4, Art. 110, the net level premium on this increasing insurance is $206.79. Hence if G denotes the gross premium, $G = 206.79 + .13\,G$. Solving, $G = \$237.69$.

EXAMPLE 2. A ten-year "term-endowment" policy provides that if the insured dies during the ten years, the beneficiary receives the face of the policy plus the premiums paid; but if the insured survives the ten-year term, he gets the amount of the premiums paid. The gross premium at age 35 on the company's regular 10-year term policy for \$1000 is \$13.10. Find the gross premium, assuming a 30 per cent loading on increasing insurance and a 15 per cent loading on 10-year pure endowment.

SOLUTION. Let G denote the gross premium. By the method in Example 2, Art. 108, the net level annual premium for a pure endowment of $10\,G$ issued at age 35 and due in 10 years is $.07765\,(10\,G)$. The gross premium for this pure endowment is then $.07765\,(10\,G)(1.15)$. By the method in Example 4, Art. 110, the net annual premium for an increasing insurance of G issued at age 35 for a term of 10 years is $.04972\,G$. The gross premium for this increasing insurance is then $.04972\,G\,(1.3)$.

Hence $G = 13.10 + .04972\;(1.3)\;G + .07765\;(1.15)\;10\,G$. Solving, $G = \$308.96$.

EXAMPLE 3. A policy issued at age 35 provides a pure endowment of \$1300 at age 65 with the return of the premiums paid in the event of death before age 65. Find the gross premium on this policy if the net level premium for the increasing insurance is loaded as in Example 1, and if the gross premium P' for the \$1300 pure endowment is given by $P' = \dfrac{P}{.87} + 1$, where P is the net level premium for this pure endowment.

SOLUTION. Let G denote the gross premium on this policy. By the result found in Example 1 the gross premium for the increasing insurance of G is $.23769\,G$. By the method in Example 2, Art. 108, the level net premium P, on the \$1300 pure endowment, is \$16.7918. The gross premium P' for this pure endowment is then $P' = \dfrac{P}{.87} + 1 = \$20.301$. Hence $G = 20.30 + .23769\,G$. Solving for G gives $G = \$26.63$.

EXERCISES

1. Show that the gross premium, G, for an ordinary whole life insurance of \$1000 issued at age x with return of all gross premiums paid prior to death is given by

$$G = \frac{1000\,M_x(1 + p) + cN_x}{N_x - R_x(1 + p)}$$

the loading being a constant c, plus p times the net annual premium P.

[HINT. Eliminate P from $P\dfrac{N_x}{D_x} = \dfrac{1000\,M_x + GR_x}{D_x}$ and $G = P(1 + p) + c$ and solve for G.]

2. Compute G in Exercise 1 for $p = .25$, $c = 1$ and $x = 25$.

3. Compute the net annual premium for an ordinary whole life insurance of \$1000 issued at age 25 with return of all net premiums paid prior to death.

Valuation of Policies, Cost of Insurance, Dividends, Policy Options

115. Reserves defined. The table of premiums in Art. 113 shows that the net level premium of a \$1000 ordinary whole life insurance policy issued at age 20 is \$13.48. The same table shows that the natural premiums are smaller than this net level premium for ages under 51 and larger for ages over 54. In other words the net level premiums of this policy provide more than is needed to pay death losses during each of the earlier years and less than is needed during each of the later years. The parts of the premiums of the early years not needed for mortality are accumulated at compound interest and held by the company to meet the heavier mortality during the later years. The amount so held at any age after the age of issue is called the reserve, or value of the policy, at that age. For example, the reserve on the above policy at the end of two years is given by

$$\frac{l_{20}(13.48 - 7.54)(1.035)^2 + l_{21}(13.48 - 7.59)(1.035)}{l_{22}} = \$12.61$$

where 7.54 and 7.59 are the natural premiums at the ages 20, 21, respectively. This method of computation becomes more tedious as the age at which the reserve is to be found becomes larger.

A simple method of computation is obtained by viewing the reserve on the policy at age $20 + n$ as the value at age $20 + n$ of the premiums paid from age 20 to age $20 + n$ less the value at age $20 + n$ of the insurance (death losses) whose term begins at age 20 and ends at age $20 + n$. These values can be written at once by use of formulas (1) and (5), Chapter IV. By this method the reserve on the \$1000 ordinary whole life policy at the end of two years is given by

$$13.48\frac{N_{20} - N_{22}}{D_{22}} - 1000\frac{M_{20} - M_{22}}{D_{22}} = \$12.61$$

Another simple method of computation is obtained by viewing the reserve on this policy as the value at age $20 + n$ of the whole life insurance whose term begins at age $20 + n$ less the value at age $20 + n$ of the premiums (13.48) paid from age $20 + n$ through

life. The value of the whole life insurance at age $20 + n$ is the net single premium at this age, and this net single premium would of itself be just enough to enable the company to pay all future death losses. The company receives, however, the premiums (13.48) from age $20 + n$ through life. It follows that the net single premium at age $20 + n$ less the value at this age of the future net annual premiums (13.48) is the reserve at age $20 + n$ on the ordinary whole life policy issued at age 20. By this method the reserve on the \$1000 ordinary whole life policy at the end of two years is given by

$$1000 \frac{M_{22}}{D_{22}} - 13.48 \frac{N_{22}}{D_{22}} = \$12.61$$

The value of any policy at the end of n years after the age of issue may be found by methods analogous to those just given for a whole life insurance policy issued at age 20. The definitions of reserve upon which the last two methods of computation are based may be stated in the general forms:

1. *The value, or terminal reserve, at age $x + n$ of a policy issued at age x is the value at age $x + n$ of the net premiums of the policy paid from age x to age $x + n$ less the value at age $x + n$ of the benefits provided by the policy from age x to age $x + n$.*
2. *The value, or terminal reserve, at age $x + n$ of a policy issued at age x is the value at age $x + n$ of the benefits provided by the policy from age $x + n$ to the end of its term less the value at age $x + n$ of the net premiums of the policy from age $x + n$ to the end of the term during which the premiums are paid.*

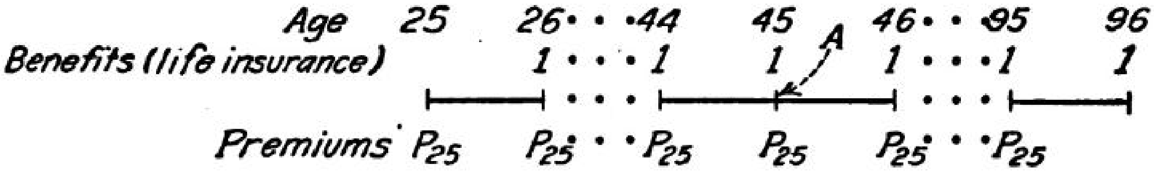
The method of computation based on the first form of definition is called the *retrospective method;* that based on the second form is called the *prospective method.* In Arts. 116 and 117 these methods are applied to find expressions for the reserves of the common forms of policies.

116. Retrospective and prospective methods of valuation. Notation. By use of the definitions in Art. 115 and the formulas for the values of life annuities and life insurances the expressions for the terminal reserve under the retrospective and the prospective

methods can be written at once. The two expressions give the same value of a given policy under a given set of net premiums, but one may be simpler to compute than the other.

EXAMPLE 1. Use the net level premiums, P_{25}, to find the terminal reserve, V, at the end of 20 years of a whole life insurance policy of face value 1 issued at age 25.

SOLUTION. The policy benefits and premiums are represented by the following diagram.



The reserve is to be found at the point A in this diagram, that is, at age 45. By the retrospective method this reserve is the value at age 45 of the premiums to the left of the vertical line through A less the value at age 45 of the benefits to the left of this line. By the prospective method this reserve is the value at age 45 of the benefits to the right of this line less the value at age 45 of the premiums to the right of it. That is, using formulas (1) and (5), Chapter IV, by the

Retrospective method: $V = P_{25}\dfrac{N_{25} - N_{45}}{D_{45}} - \dfrac{M_{25} - M_{45}}{D_{45}} = .21304$

Prospective method: $V = \dfrac{M_{45}}{D_{45}} - P_{25}\dfrac{N_{45}}{D_{45}} = .21304$

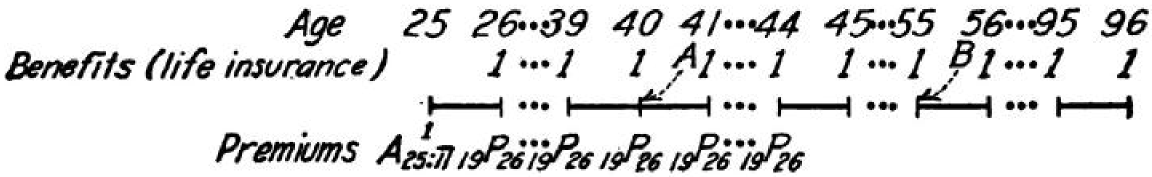
EXERCISE 1. By replacing P_{25} by its value in commutation symbols, show that the two expressions for V are equal.

EXERCISE 2. Write the expression for the reserve of this policy at age 95. (Use the prospective method.)

EXERCISE 3. Write the expression for the reserve of this policy at age 26. (Use the retrospective method.)

EXAMPLE 2. Use the full preliminary term net premiums, $A^1_{25:\overline{1}|}$ and ${}_{19}P_{26}$, to find the terminal reserves, V, at the end of 15 years and at the end of 30 years of a \$1 20-payment life insurance policy issued at age 25.

SOLUTION. The policy benefits and premiums are represented by the following diagram:



The reserves are to be found at the points A and B in this diagram. By the retrospective method the reserve at A is given by

$$V = {}_{19}P_{26}\frac{N_{26} - N_{40}}{D_{40}} - \frac{M_{26} - M_{40}}{D_{40}} = .30148$$

In this case the reserve at age 26 is zero. By the prospective method the reserve at B is given by $V = \dfrac{M_{55}}{D_{55}} = .56615.$

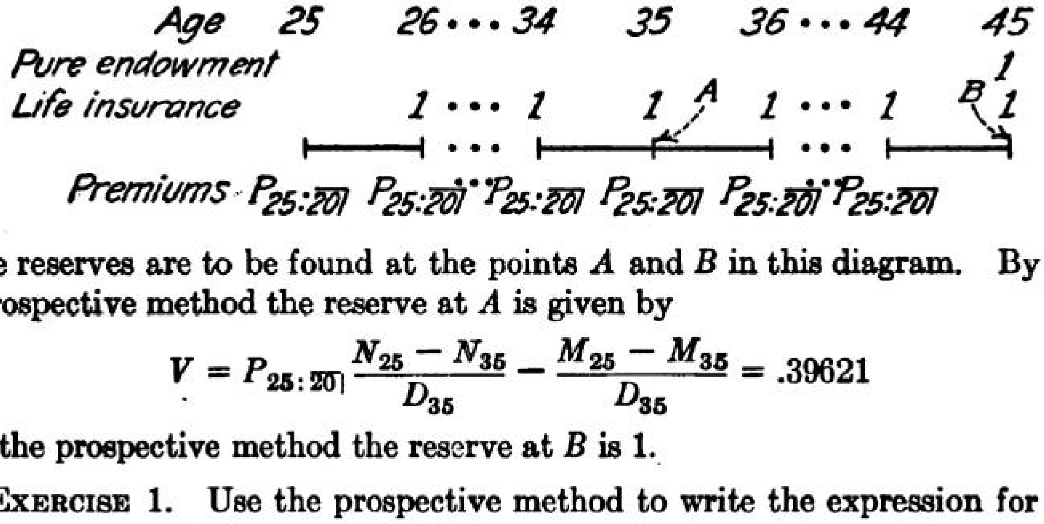
Exercise 1. Use the prospective method to write the expression for the reserve at A and show that it is the same as that found by the retrospective method.

Exercise 2. Use the retrospective method to write the expression for the reserve at B and show that it is the same as that found by the prospective method.

Exercise 3. Find the reserves at A and B on a \$1000 policy of the type in Example 2.

Example 3. Use the net level premiums, $P_{25:\overline{20|}}$, to find the terminal reserves, V, at the end of 10 years and at the end of 20 years, of a \$1 20-year endowment insurance policy issued at age 25.

Solution. The policy benefits and the premiums are represented by the following diagram:



The reserves are to be found at the points A and B in this diagram. By the retrospective method the reserve at A is given by

$$V = P_{25:\overline{20|}}\frac{N_{25} - N_{35}}{D_{35}} - \frac{M_{25} - M_{35}}{D_{35}} = .39621$$

By the prospective method the reserve at B is 1.

Exercise 1. Use the prospective method to write the expression for the reserve at A and show that it is the same as that found by the retrospective method.

Exercise 2. Use the retrospective method to write the expression for the reserve at B and show that it equals 1.

EXERCISES

1. Use net level premiums to compute the terminal reserve, at the end of 5 years, of a \$1000 10-year term insurance policy issued at age 25.

2. Use net level premiums to compute terminal reserve, at end of 10 years, of a \$1000 ordinary joint whole life insurance policy issued at ages 28, 32.

117. Reserve formulas based on net level and on full preliminary term net premiums. Notation. A general formula for the terminal reserve at age $x + n$, under a given set of premiums, can readily be obtained for any policy issued at age x by either the retrospective or the prospective method. There is a standard notation for this reserve under the net level annual premiums for the various forms of life insurance policies of face value 1 issued at age x. In what follows in this article this notation is given for the common form of policies, and the formulas for the values of the terminal reserve at age $x + n_1$, that is, at the end of the $n_1{}^{th}$ policy year, are written.

Ordinary whole life insurance policy

Retrospective method:

$$_{n_1}V_x = P_x \frac{N_x - N_{x+n_1}}{D_{x+n_1}} - \frac{M_x - M_{x+n_1}}{D_{x+n_1}}$$

Prospective method:

$$_{n_1}V_x = \frac{M_{x+n_1}}{D_{x+n_1}} - P_x \frac{N_{x+n_1}}{D_{x+n_1}}$$

m-payment whole life insurance policy

Retrospective method:

$$n_1 \leq m, \quad {}_{n_1:m}V_x = {}_mP_x \frac{N_x - N_{x+n_1}}{D_{x+n_1}} - \frac{M_x - M_{x+n_1}}{D_{x+n_1}}$$

$$n_1 \geq m, \quad = {}_mP_x \frac{N_x - N_{x+m}}{D_{x+n_1}} - \frac{M_x - M_{x+n_1}}{D_{x+n_1}}$$

Prospective method:

$$n_1 \leq m, \quad {}_{n_1:m}V_x = \frac{M_{x+n_1}}{D_{x+n_1}} - {}_mP_x \frac{N_{x+n_1} - N_{x+m}}{D_{x+n_1}}$$

$$n \geq m, \quad = \frac{M_{x+n_1}}{D_{x+n_1}}$$

n-year endowment insurance policy

Retrospective method:

$$_{n_1}V_{x\overline{n}|} = P_{x\overline{n}|} \frac{N_x - N_{x+n_1}}{D_{x+n_1}} - \frac{M_x - M_{x+n_1}}{D_{x+n_1}}$$

Prospective method:

$$ {}_{n_1}V_{x\overline{n}|} = \frac{M_{x+n_1} - M_{x+n} + D_{x+n}}{D_{x+n_1}} - P_{x\overline{n}|}\frac{N_{x+n_1} - N_{x+n}}{D_{x+n_1}} $$

It should be noted that when $n_1 = n$, ${}_nV_{x\overline{n}|} = 1$.

n-payment, n-year term insurance policy

Retrospective method:

$$ {}_{n_1}V^1_{x\overline{n}|} = P^1_{x\overline{n}|}\frac{N_x - N_{x+n_1}}{D_{x+n_1}} - \frac{M_x - M_{x+n_1}}{D_{x+n_1}} $$

Prospective method:

$$ {}_{n_1}V^1_{x\overline{n}|} = \frac{M_{x+n_1} - M_{x+n}}{D_{x+n_1}} - P^1_{x\overline{n}|}\frac{N_{x+n_1} - N_{x+n}}{D_{x+n_1}} $$

Exercise. Verify that the two methods lead to the same terminal reserve for each of the above insurance policies.

The notation for the net level premium reserves of a given policy can be used also to represent the full preliminary term premium reserves of the policy. For example, the full preliminary term premium reserve at the end of fifteen years of an ordinary whole life policy of face value 1 issued at age 30 is represented by ${}_{14}V_{31}$ and that of a 20-payment life policy is represented by ${}_{14:19}V_{31}$.

Under full preliminary term valuation the first year's insurance of a given policy is treated as a one-year term policy and the subsequent years as a policy of the same kind whose term ends at the same time as that of the given policy but begins at an age one year higher. In this method of valuation there is no reserve at the end of the first year. By the prospective method it is seen that the full preliminary term premium reserve and the net level premium reserve of a given policy are equal in value at the end of the premium payment term or at any subsequent time. The two reserves are not equal, however, at any age prior to that at the end of the premium payment term, the reserve based on the net level premiums being greater in this case than that based on the full preliminary term premiums. This follows at once by the prospective method upon noting that the full preliminary term renewal premium is greater than the net level premium. For some policies the differences between these reserves are large during the early policy years. For example, on a \$1000 10-year endowment

insurance policy issued at age 25, the net level premium reserve at the end of the second year is \$167.66 and the full preliminary term premium reserve is \$93.24. In this case the difference is \$74.42. On this policy the differences range from \$82.08 at the end of the first year to zero at the end of the tenth year. For such policies the modified preliminary term plans of valuation discussed in Arts. 120 to 123 inclusive are more satisfactory than the full preliminary term plan.

EXERCISES

1. Write the symbols which represent the net level premium reserves at the end of 1, 2, 10, and 15 years of the following policies of face value 1 issued at age 35: ordinary whole life; 25-payment life; 15-year endowment; 15-year term.

2. Same as Exercise 1 except that the net level premiums are replaced by the full preliminary term premiums.

3. Show that ${}_{n_1}V_x = A_{x+n_1} - P_x \mathbf{a}_{x+n_1} = 1 - \dfrac{\mathbf{a}_{x+n_1}}{\mathbf{a}_x}$

[HINT. Use Exercise 2, Art. 111.]

4. The following table shows the net level premium reserves and the full preliminary term premium reserves after 1, 2, 10, 15, 20, and 25 years for some \$1000 insurance policies issued at age 35; the first row for each year gives the net level premium reserves and the second row gives the full preliminary term reserves:

YEAR	ORDINARY LIFE	20-PAYMENT LIFE	10-PAYMENT LIFE	15-YEAR ENDOWMENT	20-YEAR ENDOWMENT
1	11.76 0.00	19.58 0.00	37.73 0.00	48.80 0.00	32.86 0.00
2	23.91 12.29	39.90 21.01	77.01 42.79	99.63 53.44	67.06 35.37
10	135.76 125.48	232.19 219.96	456.00 456.00	592.44 571.53	395.99 375.46
15	219.15 209.85	384.02 377.22	508.49 508.49	1000.00 1000.00	664.10 652.58
20	310.75 302.54	566.15 566.15	566.15 566.15		1000.00 1000.00
25	407.30 400.25	626.92 626.92	626.92 626.92		

Verify a few of these reserves.

118. Fackler's accumulation formula. By means of the formulas for the values of life annuities and of life insurances a formula can be written which is useful in computing tables of reserves of a given policy. The net level premium reserves at the end of n_1 years and of $n_1 + 1$ years of an ordinary whole life policy of face value 1 issued at age x are represented by ${}_{n_1}V_x$ and ${}_{n_1+1}V_x$ respectively. By the retrospective method it is seen that ${}_{n_1+1}V_x$ is the value of ${}_{n_1}V_x + P_x$ accumulated as a pure endowment from age $x + n_1$ to age $x + n_1 + 1$ less the value at age $x + n_1 + 1$ of the insurance (death losses) whose term begins at age $x + n_1$ and ends at age $x + n_1 + 1$. Hence by formulas (1_2) and (5_2), Chapter IV,

$$
\begin{aligned}
{}_{n_1+1}V_x &= ({}_{n_1}V_x + P_x)\frac{D_{x+n_1}}{D_{x+n_1+1}} - \frac{C_{x+n_1}}{D_{x+n_1+1}} \\
&= ({}_{n_1}V_x + P_x)u_{x+n_1} - k_{x+n_1}
\end{aligned}
$$

where u_{x+n_1} and k_{x+n_1} are the valuation symbols defined in Arts. 82 and 90, Chapter IV. This relation connecting ${}_{n_1+1}V_x$ and ${}_{n_1}V_x$ is called *Fackler's accumulation formula.* An analogous formula can be written for any policy. Another accumulation formula is given in Exercise 3 below.

EXERCISES

1. Compute ${}_2V_{35}$ and ${}_3V_{35}$ by the use of Fackler's Formula.

2. Show that ${}_{n_1+t}V_x = {}_{n_1}V_x \dfrac{D_{x+n_1}}{D_{x+n_1+t}} + P_x \cdot {}_tu_{x+n_1} - {}_tk_{x+n_1}$.

3. Use the relation given in Exercise 3, Art. 117, to show that

$$ {}_{n_1+1}V_x = {}_{n_1}V_n + \frac{\text{a}_{x+n_1} - \text{a}_{x+n_1+1}}{\text{a}_x}. $$

[Note that this formula is well suited for computing a table of reserves when life annuity tables are available.]

119. Cost of Insurance. If death occurs during the $n_1{}^{\text{th}}$ policy year of an ordinary whole life insurance policy of face value 1 issued at age x, $1 - {}_{n_1}V_x$ will be needed at the end of the year in addition to the net level reserve, ${}_{n_1}V_x$, at that time to pay the face value of the policy. This sum $1 - {}_{n_1}V_x$ is called the *amount at*

risk during the n_1^{th} policy year. The value of the expectation that the amount at risk will be paid at the end of the year is then $q_{x+n_1-1}\,(1 - {}_{n_1}V_x)$, where q_{x+n_1-1} is the probability that a person aged $x + n_1 - 1$ will die within one year. The value of this expectation is the mortality cost during the year. It is called the cost of insurance for the n_1^{th} policy year, and it will be represented by the symbol ${}_{n_1}K_x$. It follows that

$$ {}_{n_1}K_x = q_{x+n_1-1}\,(1 - {}_{n_1}V_x) $$

A similar formula can be written for any other insurance policy. Analogous formulas can also be written based on full preliminary term reserves and on the modified preliminary term reserves discussed in Arts. 120 to 123 inclusive.

EXAMPLE. Compute the cost of insurance during the 10th policy year of a \$1000 20-year endowment insurance policy issued at age 35.

SOLUTION. In this case the cost of insurance is given by

$$ \begin{aligned} 1000\ {}_{10}K_{35:\overline{20|}} &= 1000\ q_{44}\ (1 - {}_{10}V_{35:\overline{20|}}) \\ &= 1000\ (.010829)(.60401) \\ &= \$6.54 \end{aligned} $$

EXERCISE. Solve this example if the full preliminary term reserve is used instead of the net level reserve.

If the reserve, ${}_{n_1}V_x + P_x$, at the beginning of the $n_1 + 1^{\text{th}}$ policy year of an ordinary whole life insurance policy of face value 1 issued at age x be improved at interest for one year and the cost of insurance be deducted the difference obtained is the terminal reserve at the end of the year. This leads to the important relation,

$$ {}_{n_1+1}V_x = ({}_{n_1}V_x + P_x)(1 + i) - {}_{n_1+1}K_x $$

EXERCISE 1. Verify this relation by substituting expressions for the values of the symbols occurring in it.

EXERCISE 2. Solving

$$ {}_{n_1+1}V_x = ({}_{n_1}V_x + P_x)(1 + i) - q_{x+n_1}(1 - {}_{n_1+1}V_x) \quad \text{for} \quad {}_{n_1+1}V_x \quad \text{gives} $$

$$ {}_{n_1+1}V_x = ({}_{n_1}V_x + P_x)\,\frac{1 + i}{1 - q_{x+n_1}} - \frac{q_{x+n_1}}{1 - q_{x+n_1}} $$

Show that this is Fackler's formula in another form.

EXERCISE

The following table shows the cost of insurance under net level reserves during the 1st, 5th, 10th, 15th, 20th, and 25th years for some $1000 insurance policies issued at age 35:

Year	Ordinary Life	20-Payment Life	10-Payment Life	15-Year Endowment	20-Year Endowment
1	8.84	8.77	8.61	8.51	8.65
5	8.99	8.58	7.62	7.04	7.88
10	9.36	8.32	5.89	0.00	6.54
15	10.23	8.07	6.44		4.40
20	11.99	7.55	7.55		0.00
25	14.65	9.22	9.22		

Verify a few of these costs.

120. Modified preliminary term valuation. Ordinary whole life basis. Loadings based on net level premiums are uniform in size. They were well suited for paying policy expenses when these were spread over the whole premium payment term. They are not well suited at present, however, since much of the expense, such as agents' commissions and medical and inspection fees, occurs during the first year of insurance. Under the net level plan the first-year loading is not sufficient to cover the first-year expense of a policy and it must be supplemented from the surplus funds.

Loadings based on full preliminary term net premiums are much larger for the first year and somewhat smaller for subsequent years than those based on the net level premiums of a given policy. These loadings are not satisfactory for limited payment life and for endowment insurance policies which have short premium payment terms, since for such policies the first-year loadings are greater than what is needed for first year expenses. For example, a $1000 10-year endowment insurance policy issued at age 25 has a natural premium of $7.79 and a level net premium of $86.45. If this policy carries a gross premium of $103.39, the first-year loading under the net level plan is $16.94, while that under the full

preliminary term plan is \$95.60. This last amount is greatly in excess of what should be allowed for first-year loading.

The insurance laws of each state in the United States specify the *minimum* amount which a company doing business in the state can hold as a reserve on each policy. To remove the objection to excessive first-year loading for some policies under full preliminary term valuation certain modified preliminary term methods are used. In this article the so-called *modified preliminary term valuation* * is presented. In this method the ordinary whole life insurance policy is used as a basis for the modifications.**

If m is the number of years in the premium payment term of a given policy issued at age x, it was seen in Art. 117 that the full preliminary term premium reserve at the end of the m^{th} policy year equals the net level premium reserve at this time, but is less than the net level at the end of any earlier policy year. Under modified preliminary term valuation the reserve of this policy is brought up to the net level at the end of m years by adding to the full preliminary term reserve at the end of n_1 years, where n_1 equals $1, 2, \cdots, m$, of an ordinary whole life policy issued at age x and having the same face value as the given policy, the value at age $x + n_1$ of a life annuity due whose term begins at age x and ends at age $x + n_1$ and whose annual rent is the net annual premium of a pure endowment issued at age x and due in m years with face value equal to the net level premium reserve at age $x + m$ of the given policy less the full preliminary term reserve at this age of the ordinary whole life policy. Ordinary whole life policies and ordinary term policies are not affected by this method of valuation. If the pure endowment annual premium corresponding to any policy of face value 1 issued at age x is denoted by π_x, the first year premium under modified preliminary term valuation is $A^1_{x\overline{1}|} + \pi_x$ and each subsequent premium is $P_{x+1} + \pi_x$.

Example 1. Use modified preliminary term valuation to find the net annual premiums of a \$1000 15-year endowment insurance policy issued at age 25.

* This is sometimes called straight modified preliminary term valuation.

** The Ohio Standard uses this method for all limited payment life and endowment policies having fewer than twenty annual premiums.

SOLUTION. In this case $1000(1 - {}_{14}V_{26})$ is the difference between the net level premium reserve of the given policy at the end of 15 years and the full preliminary term reserve of the ordinary whole life policy. By Art. 109 it follows that, for this policy,

$$1000\,\pi_{25} = 1000(1 - {}_{14}V_{26})\,\frac{D_{40}}{N_{25} - N_{40}}$$
$$= \$39.83$$

Hence the required premiums are

First year: $1000\,A^{\;1}_{25\,\overline{1}|} + 39.83 = 7.79 + 39.83 = \$47.62$

Last fourteen years: $1000\,P_{26} + 39.83 = 15.48 + 39.83 = \$55.31$

EXERCISE 1. Show that, under modified preliminary term valuation, the net annual premiums of an n-year endowment insurance policy of face value 1 issued at age x are

$$\left.\begin{array}{ll}\text{First year:} & \dfrac{C_x}{D_x} + \pi_x \\ \text{Last } n-1 \text{ years:} & P_{x+1} + \pi_x\end{array}\right\}\text{ where } \pi_x = \frac{1 - {}_{n-1}V_{x+1}}{{}_n u_x}$$

EXERCISE 2. Show that, under modified preliminary term valuation, the net annual premiums of an m-payment, n-year endowment insurance policy of face value 1 issued at age x are

$$\left.\begin{array}{ll}\text{First year:} & \dfrac{C_x}{D_x} + \pi_x \\ \text{Last } m-1 \text{ years:} & P_{x+1} + \pi_x\end{array}\right\}\text{ where } \pi_x = \frac{A_{x+m:\overline{n-m}|} - {}_{m-1}V_{x+1}}{{}_m u_x}$$

EXERCISE 3. Express the values of π_x in Exercises 1 and 2 in commutation symbols.

EXAMPLE 2. Use modified preliminary term valuation to find the net annual premiums of a \$1000 10-payment whole life insurance policy issued at age 25.

SOLUTION. In this case $1000(A_{35} - {}_9V_{26})$ is the difference between the net level premium reserve of the given policy at the end of 10 years and the full preliminary term reserve of the ordinary whole life policy at this time. By Art. 109 it follows, that for this policy,

$$1000\,\pi_{25} = 1000(A_{35} - {}_9V_{26})\,\frac{D_{35}}{N_{25} - N_{35}}$$
$$= \$22.58$$

Hence, the required premiums are

First year: $1000\,A^{\;1}_{25\,\overline{1}|} + 22.58 = 7.79 + 22.58 = \$30.37$

Last nine years: $1000\,P_{26} + 22.58 = 15.48 + 22.58 = \$38.06$

EXERCISE 4. Show that, under modified preliminary term valuation, the net annual premiums of an m-payment whole life insurance policy issued at age x are

$$\left.\begin{array}{ll}\text{First year:} & \dfrac{C_x}{D_x} + \pi_x \\ \text{Last } m-1 \text{ years:} & P_{x+1} + \pi_x\end{array}\right\}\text{ where } \pi_x = \frac{A_{x+m} - {}_{m-1}V_{x+1}}{{}_m u_x}$$

EXERCISE 5. Express π_x in Exercise 4 in commutation symbols.

In computing reserves under modified preliminary term valuation, it is convenient to resolve the net premiums into the two sets of which they are composed, viz.: the pure endowment premiums, and the full preliminary term ordinary whole life premiums. Tables of net level premium reserves can then be used in the computations.

EXAMPLE 3. Compute the terminal reserves under modified preliminary term valuation after 1, 5, and 10 years of a \$1000 10-payment life insurance policy issued at age 25.

SOLUTION. The net premiums for this policy are given in the solution of Example 2. Upon resolving these premiums into the two component sets used in finding them, it is seen by the retrospective method that the terminal reserves to be found are

First year: $22.58\ u_{25} = \$23.55$
Fifth year: $1000\ {}_4V_{26} + 22.58\ {}_5u_{25} = 33.54 + 128.55 = \$162.09$
Tenth year: $1000\ {}_9V_{26} + 22.58\ {}_{10}u_{25} = 82.42 + 288.16 = \$370.58$

EXERCISE 6. Solve Example 3 by use of the prospective method of computing reserves. [Note that the reserve after 10 years is the net single premium, A_{35}.]

EXERCISES

1. The following table shows the net premiums under modified preliminary term valuation of some \$1000 insurance policies issued at age 35:

YEAR	20-PAYMENT LIFE	10-PAYMENT LIFE	15-YEAR ENDOWMENT	20-YEAR ENDOWMENT
First . . .	16.38	34.32	44.54	29.10
Subsequent .	28.28	46.22	56.44	41.00

Verify the premiums for the 20-year endowment insurance policy.

2. Show that, for an m-payment life insurance policy of face value 1 issued at age x, the terminal reserve under modified preliminary term valuation after

1 year is $\pi_x u_x$
t years $(1 < t < m)$ is ${}_{t-1}V_{x+1} + \pi_x\ {}_tu_x$
t years $(t \geq m)$ is A_{x+t}

where π_x is given in Exercise 4 under Example 2.

3. Show that, for an n-year endowment insurance policy of face value 1 issued at age x, the terminal reserve under modified preliminary term valuation after

1 year is $\pi_x u_x$
t years $(1 < t < n)$ is ${}_{t-1}V_{x+1} + \pi_x\ {}_tu_x$
n years is 1

where π_x is given in Exercise 1 under Example 1.

Hence the required premiums are

First year: $1000\,A^{1}_{25:\overline{1}|} + 32.35 = 7.79 + 32.35 = \$40.14$
Last fourteen years: $1000\,{}_{19}P_{26} + 32.35 = 23.68 + 32.35 = \$56.03$

EXERCISE 1. Show that when $n \leq 20$, the net annual premiums under the Illinois Standard of an n-year endowment insurance policy of face value 1 issued at age x are

$$\left.\begin{array}{ll}\text{First year:} & \dfrac{C_x}{D_x} + \pi_x \\ \text{Last } n-1 \text{ years:} & {}_{n-1}P_{x+1} + \pi_x\end{array}\right\} \text{where } \pi_x = \frac{1 - {}_{n-1:19}V_{x+1}}{{}_nu_x}$$

EXERCISE 2. Show that when $m \leq 20$, the net annual premiums under the Illinois Standard of an m-payment n-year endowment insurance policy of face value 1 issued at age x are

$$\left.\begin{array}{ll}\text{First year:} & \dfrac{C_x}{D_x} + \pi_x \\ \text{Last } m-1 \text{ years:} & {}_{19}P_{x+1} + \pi_x\end{array}\right\} \text{where } \pi_x = \frac{A_{x+m:\overline{n-m}|} - {}_{m-1:19}V_{x+1}}{{}_mu_x}$$

EXAMPLE 2. Find the net annual premiums under the Illinois Standard of a \$1000 30-year endowment insurance policy issued at age 30.

SOLUTION. In this case, $1000({}_{20}V_{30:\overline{30}|} - A_{50})$ is the difference between the net level premium reserve of the given policy at the end of 20 years and the full preliminary term reserve of the 20-payment life policy at this time. By Art. 109 it follows that, for this policy,

$$\begin{aligned} 1000\,\pi_{30} &= 1000({}_{20}V_{30:\overline{30}|} - A_{50})\frac{D_{50}}{N_{30} - N_{50}} \\ &= \$0.50 \end{aligned}$$

Hence the required premiums are

First year: $1000\,A^{1}_{30\,\overline{1}|} + 0.50 = 8.14 + 0.50 = \$8.64$
Next nineteen years: $1000\,{}_{19}P_{31} + 0.50 = 26.02 + 0.50 = \$26.52$
Last ten years: $1000\,P_{30:\overline{30}|} = \$25.21$

EXERCISE 3. Show that when $n > 20$, the net annual premiums of an n-year endowment insurance policy of face value 1 issued at age x, which, under the Illinois Standard, is valued on a 20-payment life basis are

$$\left.\begin{array}{ll}\text{First year:} & \dfrac{C_x}{D_x} + \pi_x \\ \text{Next nineteen years:} & {}_{19}P_{x+1} + \pi_x\end{array}\right\} \text{where } \pi_x = \frac{{}_{20}V_{x\overline{n}|} - A_{x+20}}{{}_{20}u_x}$$
$$\text{Last } n - 20 \text{ years:} \quad P_{x\overline{n}|}$$

EXERCISE 4. When $\pi_x = 0$ in Exercise 3, show that $P_{x\overline{n}|} = {}_{20}P_x$. In this case the net level n-year endowment insurance premium at age x equals the net level 20-payment life insurance premium at age x.

EXAMPLE 3. Find the net annual premiums under the Illinois Standard of a \$1000 10-payment life insurance policy issued at age 25.

SOLUTION. In this case, $1000\,(A_{35} - {}_{9:19}V_{26})$ is the difference between the net level premium reserve of the given policy at the end of 10 years and the full preliminary term reserve of the 20-payment life policy at this time. By Art. 109 it follows that for this policy

$$1000\,\pi_{25} = 1000\,(A_{35} - {}_{9:19}V_{26})\frac{D_{35}}{N_{25} - N_{35}}$$
$$= \$15.36$$

Hence the required premiums are

First year: $1000\,A^{1}_{25:\overline{1|}} + 15.36 = 7.79 + 15.36 = \$23.15$

Last nine years: $1000\,{}_{19}P_{26} + 15.36 = 23.68 + 15.36 = \$39.04$

EXERCISE 5. Show that when $m < 20$, the net annual premiums under the Illinois Standard of an m-payment life insurance policy of face value 1 issued at age x are

$$\left.\begin{array}{ll}\text{First year:} & \dfrac{C_x}{D_x} + \pi_x \\ \text{Last } m-1 \text{ years:} & {}_{19}P_{x+1} + \pi_x\end{array}\right\} \text{where } \pi_x = \frac{A_{x+m} - {}_{m-1:19}V_{x+1}}{{}_{m}u_x}$$

EXAMPLE 4. Compute the terminal reserves under the Illinois Standard after 1, 8, and 15 years of a \$1000 15-year endowment insurance policy issued at age 25.

SOLUTION. The net annual premiums for this policy are given in the solution of Example 1. Upon resolving these premiums into the two component sets used in finding them it is seen by the retrospective method that the terminal reserves to be found are

First year: $32.35\,u_{25} = \$33.76$

Eighth year: $1000\,{}_{7:19}V_{26} + 32.35\,{}_{8}u_{25} = 130.32 + 315.34 = \$445.66$

Fifteenth year: $1000\,{}_{14:19}V_{26} + 32.35\,{}_{15}u_{25} = 301.48 + 691.51 = \$999.99$

EXERCISE 6. Solve Example 4 by use of the prospective method of computing reserves. [Note that the reserve after 15 years is \$1000.]

EXERCISES

1. The following table shows the net annual premiums under the Illinois Standard of some \$1000 policies issued at age 35:

YEAR	10-PAYMENT LIFE	15-PAYMENT LIFE	15-YEAR ENDOWMENT	20-YEAR ENDOWMENT	25-YEAR ENDOWMENT	15-PAYMENT ENDOWMENT AT 65
First . .	26.97	14.61	36.93	21.36	12.75	20.58
Subsequent	47.22	34.85	57.18	41.61	(2 to 20) \$33.00 (21 to 25) \$31.50	40.83

Verify the premiums for the 25-year endowment and the 15-payment endowment at 65.

2. Show that, under the Illinois Standard, the net annual premiums of a 25-year endowment insurance policy are full preliminary term when the age of issue is equal to or greater than 46, and modified preliminary term (20-payment life basis) when the age of issue is less than 46. [See Fackler and Fackler, Illinois Standard Tables.]

3. Show that, under the Illinois Standard, the net premiums of an endowment insurance at 65 are full preliminary term when the age of issue is equal to or less than 37, and modified preliminary term when the age of issue is greater than 37.

4. Show that for an m-payment ($m < 20$), whole life insurance policy of ace value 1 issued at age x, the terminal reserve under the Illinois Standard after

$$\begin{aligned} &1 \text{ year is } \pi_x u_x \\ &t \text{ years } (1 < t < m) \text{ is } {}_{t-1:19}V_{x+1} + \pi_{x\,t}u_x \\ &t \text{ years } (t \geq m) \text{ is } A_{x+t} \end{aligned}$$

where π_x is given in Exercise 5, Example 3.

5. If the m-payment life policy in Exercise 4 is replaced by an n-year endowment insurance policy ($n \leq 20$), write the corresponding expressions for the terminal reserves. [See Exercise 1, Example 1 for π_x.]

6. Same as Exercise 5 for an m-payment n-year endowment insurance policy, ($m \leq 20$). [See Exercise 2, Example 1 for π_x.]

7. The following table shows the terminal reserves after 1, 2, 5, 15, 20, and 25 years under the Illinois Standard of some $1000 policies issued at age 35.

Year	10-Payment Life	15-Payment Life	15-Year Endowment	20-Year Endowment	25-Year Endowment	15-Payment Endowment at 65
1	19.14	6.23	29.54	13.28	4.29	12.46
2	60.15	33.74	81.40	48.17	29.78	46.49
5	193.59	122.93	250.46	161.54	112.34	157.06
10	456.00	296.73	584.19	383.76	272.86	373.66
15	509.49	508.49	1000.00	657.29	467.68	640.03
20	566.15	566.15		1000.00	706.27	735.05
25	626.92	626.92			1000.00	850.55

Verify a few of these reserves.

122. New Jersey Standard. The New Jersey Standard is the same as the Illinois Standard for a limited payment life and for an

endowment insurance policy whose full preliminary term renewal premiums are greater than the full preliminary term renewal premiums of a 20-payment life policy issued at the same age and having the same face value. These policies are valued on the basis of a 20-payment life preliminary term policy by the method described in Art. 121.

Under the New Jersey Standard any policy whose full preliminary term renewal premiums are less than the full preliminary term renewal premiums of a 20-payment life policy issued at the same age and having the same face value and for which the premium charged for the first year's insurance is greater than 150% of the net premium (natural) for this year's insurance, is valued as follows: The terminal reserve of any such policy issued at age x is brought up to the net level premium reserve at the end of twenty years by adding to the full preliminary term premium reserve at the end of n_1 years, where $n_1 = 2, 3, \ldots, 20$, of the given policy the value at age $x + n_1$ of a life annuity due whose term begins at age $x + 1$ and ends at age $x + n_1$ and whose annual rent is the net annual premium of a pure endowment issued at age $x + 1$ and due in 19 years with face value equal to the net level premium reserve at age $x + 20$ of the given policy less its full preliminary term premium reserve at this age. These policies have no reserve at the end of the first year. This method applies to limited payment life and to endowment insurance policies which are not valued on the 20-payment life basis as well as to ordinary whole life policies. In what follows in this article π_{x+1} will denote the annual pure endowment premium which corresponds to any policy of face value 1.

Policies, other than industrial, which do not come under either of the two methods described may be valued, under the New Jersey Standard, on the full preliminary term plan. As stated in Art. 121, the term policies commonly issued by companies are usually valued, however, on the net level premium basis.

Example 1. Find the net annual premiums, under the New Jersey Standard, of a \$1000 ordinary whole life policy issued at age 25.

Solution. In this case, 1000 $({}_{20}V_{25} - {}_{19}V_{26})$ is the difference between the net level premium reserve and the full preliminary term premium reserve of

the given policy at the end of 20 years. By Art. 109 it follows that, for this policy,

$$1000\,\pi_{26} = 1000({}_{20}V_{25} - {}_{19}V_{26})\,\frac{D_{45}}{N_{26} - N_{45}} = \$0.20$$

Hence the required premiums are

First year: $1000\,A^{\;1}_{25:\overline{1|}} = \$7.79$

Next nineteen years: $1000\,P_{26} + 0.20 = 15.48 + 0.20 = \$15.68$

After twenty years: $1000\,P_{25} = \$15.10$

EXERCISE 1. Show that, under the New Jersey Standard, the net annual premiums of an ordinary whole life policy of face value 1 issued at age x are

First year: $\dfrac{C_x}{D_x}$

Next nineteen years: $P_{x+1} + \pi_x$ where $\pi_x = ({}_{20}V_x - {}_{19}V_{x+1})\,\dfrac{D_{x+20}}{N_{x+1} - N_{x+20}}$

After twenty years: P_x

EXAMPLE 2. Find the net annual premiums, under the New Jersey Standard, of a \$1000 25-payment life insurance policy issued at age 25.

SOLUTION. In this case, $1000\,({}_{20:25}V_{25} - {}_{19:24}V_{26})$ is the difference between the net level premium reserve and the full preliminary term premium reserve of the given policy at the end of 20 years. By Art. 109, it follows that, for this policy

$$1000\,\pi_{26} = 1000({}_{20:25}V_{25} - {}_{19:24}V_{26})\,\frac{D_{45}}{N_{26} - N_{45}} = \$0.13$$

Hence the required premiums are

First year: $1000\,A^{\;1}_{25\overline{1|}} = \$7.79$

Next nineteen years: $1000\,{}_{24}P_{26} + 0.13 = 20.58 + 0.13 = \$20.71$

Last five years: $1000\,{}_{25}P_{25} = \$19.77$

EXERCISE 2. Show that, under the New Jersey Standard, the net annual premiums of an m-payment ($m > 20$) life insurance policy of face value 1 issued at age x are

First year: $\dfrac{C_x}{D_x}$

Next nineteen years: ${}_{m-1}P_{x+1} + \pi_{x+1}$

where $$\pi_{x+1} = ({}_{20:m}V_x - {}_{19:m-1}V_{x+1})\,\frac{D_{x+20}}{N_{x+1} - N_{x+20}}$$

After twenty years: ${}_mP_x$

EXAMPLE 3. Find the net annual premiums, under the New Jersey Standard, of a \$1000 30-year endowment insurance policy issued at age 35.

SOLUTION. In this case, π_{36} is given by

$$1000\,\pi_{36} = 1000\,({}_{20}V_{35:\overline{30|}} - {}_{19}V_{36:\overline{29|}})\,\frac{D_{55}}{N_{36} - N_{55}} = \$0.27$$

and the required premiums are

First year: $1000\,A^{\;1}_{35:\overline{1|}} = \$8.64$

Next nineteen years: $1000\,P_{36:\overline{29|}} + 0.27 = \$27.44 + 0.27 = \$27.71$

Last ten years: $1000\,P_{35:\overline{30|}} = \$26.31$

EXERCISE 3. Show that, under the New Jersey Standard, the net annual premiums of an n-year endowment insurance policy of face value 1 issued at age x which is valued on a full preliminary policy are

First year: $\dfrac{C_x}{D_x}$

Next nineteen years: $P_{x+1:\overline{n-1|}} + \pi_{x+1}$

where $$\pi_{x+1} = ({}_{20}V_{x\overline{n|}} - {}_{19}V_{x+1:\overline{n-1|}})\frac{D_{x+20}}{N_{x+1} - N_{x+20}}$$

Last $n - 20$ years: $P_{x\overline{n|}}$

EXAMPLE 4. Compute the terminal reserves under the New Jersey Standard, after 1, 2, 8, and 25 years of a \$1000 ordinary whole life policy issued at age 25.

SOLUTION. The net annual premiums for this policy are given in the solution of Example 1. Since the first premium is the natural premium there is no reserve at the end of one year. Using the retrospective method for the reserves after two and eight years and the prospective method for the reserves after 25 years, the reserves are found to be

Second year: $1000\,{}_1V_{26} + .20\,\dfrac{D_{25}}{D_{26}} = \$8.17$

Eighth year: $1000\,{}_7V_{26} + .20\,\dfrac{N_{26} - N_{33}}{D_{33}} = \$63.55$

Twenty-fifth year: $1000\,A_{50} - 15.10\,\dfrac{N_{50}}{D_{50}} = \$288.97$

EXERCISES

1. The following table shows the net annual premiums under the New Jersey Standard of some \$1000 policies issued at age 35.

YEAR	ORDINARY WHOLE LIFE	30-PAYMENT LIFE	20-PAYMENT LIFE	20-PAYMENT ENDOWMENT AT 65
First	8.64	8.64	8.64	13.59
Next nineteen . .	20.81	23.37	28.89	33.84
After twenty . . .	19.91	22.28		

Verify the premiums for the 30-payment life and the 20-payment endowment at 65.

2. Show that for an ordinary whole life insurance policy of face value 1 issued at age x the terminal reserve under the New Jersey Standard after

1 year is 0

t years $(1 < t < 20)$ is ${}_{t-1}V_{x+1} + \pi_{x+1}({}_{t-1}u_{x+1})$

t years $(t \geq 20)$ is $A_{x+t} - P_x\dfrac{N_{x+t}}{D_{x+t}}$

where π_{x+1} is given in Exercise 1, Example 1.

3. If the ordinary whole life policy in Exercise 2 is replaced by an m-payment life policy ($m > 20$), write the corresponding expressions for the terminal reserves. [See Exercise 2, Example 2 for π_{x+1}.]

4. Same as Exercise 3 for an n-year endowment insurance policy which, under the New Jersey Standard, is valued on a full preliminary term n-year endowment insurance policy. [See Exercise 3, Example 3 for π_{x+1}.]

5. The following table shows the terminal reserves after 1, 2, 5, 10, 20, and 25 years under the New Jersey Standard of some \$1000 policies issued at age 35:

Year	Ordinary Whole Life	30-Payment Life	20-Payment Life	20-Payment Endowment at 65
1	0	0	0	5.17
2	12.56	15.24	21.01	31.58
5	52.74	64.18	88.88	117.16
10	128.42	157.40	219.96	283.73
15	215.06	266.40	377.22	486.26
20	310.75	391.61	566.15	723.05
25	407.30	528.47	626.92	850.55

Verify a few of these reserves.

123. Other state standards. Select and ultimate valuation. By the use of the methods presented in the preceding articles the net premiums and the terminal reserves of policies can be determined under the standards of valuation adopted by most of the states in the United States. New York State, however, is an exception. In this state the so-called *select* and *ultimate method of valuation* is in use.* In this method, use is made of the principle that a company is benefited through the addition of new policy holders because of the light mortality among them for several years. This light mortality is due to selection based on careful medical examination. This method requires companies to use certain specified percentages of mortality shown by the American Experience Table in finding the minimum values of a policy during

* Insurance companies operating in New York may, at their option, use the Illinois Standard of valuation or the select and ultimate method of valuation.

the first five years of insurance. These percentages range from 50% for the first year to 95% for the fifth year.

For the study of the standard of valuation used in a given state, a copy of the law relating to the valuation of policies in the state should be obtained.

124. Initial and mean reserves. The terminal reserve of a policy for a given policy year is the value of the policy at the end of this year. This reserve does not include the net premium payable at the beginning of the following year. The value of the policy at the beginning of a given policy year, just after a premium is paid, is called the *initial reserve* for this year. The initial reserve of a policy for its n^{th} policy year is then the terminal reserve of the policy after $n - 1$ years plus the net premium payable at the beginning of the n^{th} year. The initial reserve of a policy for a year subsequent to the premium payment term is evidently the same as the terminal reserve for the preceding year. If the value of an ordinary whole life policy of face value 1 issued at age x at the beginning of its n^{th} policy year is represented by ${}_nI_x$, then

$$ {}_nI_x = {}_{n-1}V_x + P_x $$

The definition of initial reserve leads to a similar formula for any other policy.

In submitting annual reports to state departments of insurance, each company is required to value its policies at a specified date, which is usually December 31st. Since policy years do not usually end December 31st, terminal reserves are not applicable in finding values at this date. The value of any policy at the end of a calendar year might be found by the use of a formula which would give its value $n + k$ years $(k < 1)$ after its date of issue. This process would involve tedious computations, however. In America, it is customary to use the average of initial and terminal reserves in finding the values of policies at the end of a calendar year. This practice is based on the assumption that the dates of issue of policies issued during any calendar year are distributed uniformly throughout the year, so that on the average they are issued about the middle of the year. The value of a policy obtained by taking the arithmetic mean of its initial and terminal n^{th} year

reserves is called the *mean reserve* of the policy for the n^{th} policy year. The mean reserve for the n^{th} year of an ordinary whole life policy of face value 1 issued at age x is given by

$$\text{Mean Reserve} = \frac{{}_nI_x + {}_nV_x}{2} = \frac{{}_{n-1}V_x + P_x + {}_nV_x}{2}$$

A similar formula can be written for any policy.

EXERCISES

1. Find the mean reserves for the tenth policy year under the Illinois Standard of the policies in Exercises 1 and 7, Art. 121.

2. Same as Exercise 1 for the policies in Exercises 1 and 5, Art. 122.

3. Same as Exercise 1 for the policies in Exercises 1 and 4, Art. 120.

4. Same as Exercise 1 for the policies in Exercise 3, Art. 117.

125. Surplus and dividends. The surplus provided by the gross premiums (see Art. 107) is derived principally from the following sources:

1. Interest earned in excess of that provided by the rate used in computing premiums and reserves,
2. Mortality savings,
3. Loadings in excess of expenses.

As stated in Art. 113 in stock insurance companies surplus is distributed to the stockholders in the form of dividends on the stock; in mutual insurance companies the surplus is returned to the policy holders; in mixed companies, part of the surplus goes to the stockholders and the rest goes to the policy holders. A method in common use in America of distributing surplus among the policy holders is the contribution plan. This plan was first introduced in 1863 by Sheppard Homans, actuary of the Mutual Life Insurance Company of New York. The dividend formula he introduced can be derived readily from the relation given in Art. 119 which expresses the terminal reserve for the $(n_1 + 1)^{\text{th}}$ policy year in terms of that for the $n_1{}^{\text{th}}$ year and the cost of insurance. This relation for an ordinary life policy can be written in the form,

$$({}_{n_1}V_x + P_x)(1 + i) - {}_{n_1+1}V_x - \frac{d_{x+n_1}}{l_{x+n_1}}(1 - {}_{n_1+1}V_x) = 0$$

This relation is based on the theoretical assumptions that an interest rate, i, will be earned and that the mortality table used will give the mortality accurately. In practice, however, the interest rate earned by a company is larger than that assumed in computing premiums and reserves, the actual mortality is more favorable than that given by the mortality table, and the gross premium less the necessary expense is usually greater than the net premium. It follows that if the actual experience of a company is used in place of the theoretical assumptions, the left-hand member of the above relation becomes positive and represents the contribution to surplus during the $(n_1 + 1)^{th}$ policy year. Let l'_{x+n_1}, d'_{x+n_1}, and i' represent a company's own experience and let e denote the proportionate expense of the policy for its $(n_1 + 1)^{th}$ year. Replacing l_{x+n_1}, d_{x+n_1}, i, and P_x in the left-hand member of the above relation by l'_{x+n_1}, d'_{x+n_1}, i', and $G_x - e$ respectively, where G_x is the gross premium received by the company, gives

$$\textit{Contribution to surplus} = {}_{n_1}V_x(1 + i') + (G_x - e)(1 + i') - {}_{n_1+1}V_x - \frac{d'_{x+n_1}}{l'_{x+n_1}}(1 - {}_{n_1+1}V_x)$$

This is Mr. Homans' formula as first introduced. If l denotes the loading, so that $G_x = P_x + l$, the formula can be written

$$\textit{Contribution to surplus} = [({}_{n_1}V_x + P_x)(1 + i') - {}_{n_1+1}V_x] - \frac{d'_{x+n_1}}{l'_{x+n_1}}(1 - {}_{n_1+1}V_x) + (l - e)(1 + i')$$

The three component parts of this formula correspond to the three sources of surplus mentioned at the beginning of this article. An analogous formula could be written for any policy.

To apply formulas of the type just discussed a participating company would need to use a mortality table based on its own experience as well as the actual interest rate earned. Most companies do not apply such formulas now, but they use in some form or other the three sources of divisible surplus exhibited by these formulas.* The forms in which these sources are used by com-

* For an excellent presentation of methods of surplus distribution actually in use see a paper in Vol. XI, Part I, No. 23, June 1922 of the Record of the American Institute of Actuaries by J. Charles Rietz, actuary of the Midland Mutual Life Insurance Company, Columbus, Ohio.

panies show wide variations. The interest element in a dividend formula is often determined as that percentage of the reserve which equals the difference between the interest rate a company is likely to earn over a period of years and that used in computing reserves. Some companies which determine the interest element in this way use initial reserves, some use terminal reserves, and some use mean reserves. The mortality element in the dividend formula is often determined as a percentage of the cost of insurance, the percentages used being graded usually to correspond to the age. The loading element is often determined by deducting an expense charge from the loading, the expense charge usually being a percentage of the gross premium plus a constant. For further information regarding the ways in which the chief sources of surplus are used in dividend formulas and for information regarding the actual constants used in these formulas reference may be made to the paper by Mr. J. Charles Rietz cited above. In this paper the dividend formulas of thirty companies are given.

126. Policy options. Cash surrender values, paid-up insurance, extended insurance. When a life insurance policy matures, the proceeds are paid to the beneficiary in a manner specified in the policy. In case of whole life and endowment insurance policies the proceeds are paid in cash or in some equivalent form. The equivalent form of settlement can be selected from those specified in the policy by the insured at any time during the term of the policy, or by the beneficiary upon death of the insured in case the insured has made no selection. These optional methods of settlement ordinarily provide for instalments which form annuities certain, life annuities, or both annuities certain and life annuities whose rent is payable once or more annually. In some of the options the whole cash value is paid in the form of equivalent instalments; in others, part is paid in cash and the rest in instalments. The unknowns in these optional methods of settlement can be readily computed by use of the formulas for the values of annuities and of insurances. Besides the ordinary forms of policies there are special forms each of which provides a particular kind of settlement. These special forms include the retirement annuities, the guaranteed income, and the child and educational endowments.

At the end of each year during the term of a life insurance policy, the policy has a cash or surrender value which is equal to the terminal reserve less a surrender charge whose maximum amount is regulated by state law. The surrender charge actually made varies with different companies, but it is usually small; often there is no charge at all. At any time when a policy is in force the insured has the privilege of obtaining a loan on it for an amount which may be equal or nearly equal to the cash value at the time. If the insured allows a policy to lapse, the cash value less any indebtedness is payable to him in cash or in some equivalent form. Two common equivalent forms are *extended insurance* and *paid-up insurance*. Under extended insurance, the one who surrenders his policy is insured during a stated term for an amount which equals the face of the policy, or the face of the policy less the indebtedness. Under paid-up insurance, the one who surrenders his policy is insured for a stated amount during the remainder of the term of the policy he surrenders. Policies usually specify the cash and loan values, the amount of paid-up insurance, and the term of the extended insurance for each year during the term of the policy. The term in extended insurance and the face value in paid-up insurance are easily determined by the use of the formula for the value of a life insurance.

Example 1. Under the New Jersey Standard, the terminal reserve at the end of 25 years of a \$1000 ordinary whole life policy issued at age 35 is \$407.30. If the full amount of this reserve is allowed as a cash surrender value, how much paid-up insurance will it purchase?

Solution. Substitution into the formula, $V = F\dfrac{M_x}{D_x}$, for the value at age x of a whole life insurance issued at age x gives

$$407.30 = F(.62692)$$

Solving, $$F = \$649.68$$

Example 2. If the cash surrender value at the end of 25 years of the policy in Example 1 is used to purchase extended insurance of \$1000 face value, find the term of the insurance.

Solution. Substitution into the formula for the value at age x of an n-year term insurance issued at age x, gives

$$407.30 = 1000\,\frac{M_{60} - M_{60+n}}{D_{60}}$$

Solving, $$M_{60+n} = M_{60} - .4703\,D_{60}$$
$$= 1151.435$$

By interpolation in Table XIII,

$$60 + n = 77 \text{ yrs. } 195 \text{ days}$$
$$n = 17 \text{ yrs. } 195 \text{ days.}$$

EXAMPLE 3. Solve Example 2 if there is a surrender charge of \$5.

SOLUTION. In this case \$402.30 is the net single premium of the extended insurance and the equation in n becomes,

$$402.30 = \frac{M_{60} - M_{60+n}}{D_{60}}$$

Solving,
$$M_{60+n} = M_{60} - .4023\, D_{60}$$
$$= 1651.357$$

By interpolation,
$$60 + n = 74 \text{ yrs. } 249 \text{ days,}$$
$$n = 14 \text{ yrs. } 249 \text{ days.}$$

EXAMPLE 4. Assuming that the policy in Example 1 is surrendered at the end of 25 years, and that it carries a loan of \$100 at that time, and allowing a surrender charge of \$5, find the term of extended insurance of \$900 face value. [Note that in this case the face of the policy is reduced by the amount of the loan. This is common practice. Discuss reason.]

SOLUTION. In this example the amount available for the purchase of extended insurance is \$302.30 and the equation in n becomes,

$$302.30 = 900\,\frac{M_{60} - M_{60+n}}{D_{60}}$$

Solving,
$$M_{60+n} = M_{60} - \frac{302.3\, D_{60}}{900}$$
$$= 2139.588$$

By interpolation,
$$60 + n = 72 \text{ yrs. } 71 \text{ days.}$$
$$n = 12 \text{ yrs. } 71 \text{ days.}$$

EXAMPLE 5. Under the New Jersey Standard the terminal reserve at the end of 10 years of a \$1000 20-year endowment insurance policy issued at age 35 is \$383.75. If the policy is surrendered at this time, and this reserve less \$5 is used to purchase a \$1000 10-year term insurance and a pure endowment of face value F due in 10 years, find F.

SOLUTION. The net single premium of a \$1000 10-year term insurance issued at age 55 is \$106.65. It follows that $383.75 - 106.65 - 5 = \$272.10$ is the net single premium of the pure endowment. Substitution into $V = F\dfrac{D_{x+n}}{D_x}$, gives

$$272.10 = F\,\frac{D_{55}}{D_{45}}$$

Solving,
$$F = 272.1\,\frac{D_{45}}{D_{55}}$$
$$= \$440.96$$

EXERCISES

1. Solve Example 1 under net level reserves.
2. Solve Example 2 under net level reserves.
3. Solve Example 3 under net level reserves.
4. Solve Example 4 under net level reserves.
5. Solve Example 5 under net level reserves.
6. If the policy in Example 5 is surrendered at the end of 4 years and if the cash value at that time is \$122.23, find the term of the extended insurance of \$1000 face value that can be purchased if a surrender charge of \$5 is made.
7. Find the face value of the paid-up insurance, term 14 years, that can be purchased for the cash value of the policy in Exercise 6.

Life Estates, Remainders, Inheritance Taxes

127. Life estates and remainders. Bequests are often made such that the whole or a part of the income from an estate goes to a given person during his life. The income received by such a person is called a *life estate*. Other bequests leave the whole or a part of the income from an estate to a person during a part of his life. More generally, there are bequests which leave the income from an estate to two or more persons for all or part of their lives. At the end of the terms during which the incomes are bequeathed the estate is given to one or more parties. A party who is bequeathed an estate after its income has been enjoyed by one or more persons holds what is called a *remainder* in the estate. In settling estates of this sort it is usually necessary to find the present values of the whole and the temporary life estates and also of the remainders. The values of the life estates are found by the use of the formulas for the values of annuities. The values of the remainders considered in this article can be found by the use of the formulas for the values of insurances. According to the laws of some states, however, the values of the remainders are found by deducting the values of the life estates from the appraised value of the estate. In the solutions of the examples in this article, the American Experience table at 5% is used for one life and Hunter's Makehamized American Experience table at 5% is used for two lives (Art. 104, Chapter IV). The answers are given to the nearest dollar.

EXAMPLE 1. By the terms of a will the income at 5% annually of a \$10000 estate goes to a widow aged 50 during her life. Find the value of her life estate.

SOLUTION. The value, V, of her life estate is given by

$$\begin{aligned} V &= 500\, a_{50} \\ &= 500\,(11.66175) \\ &= \$5831 \end{aligned}$$

EXAMPLE 2. By the terms of the will in Example 1, the estate goes to a hospital when the widow dies. Find the value of this remainder.

SOLUTION 1. In this solution the value V is found by use of the insurance formula

$$\begin{aligned} V &= 10000\, A_{50} \\ &= 10000\,(.3970598) \\ &= \$3971 \end{aligned}$$

SOLUTION 2. In this solution the value V is found by deducting the result found in Example 1 from \$10000. That is

$$\begin{aligned} V &= 10000 - 5830.88 \\ &= \$4169 \end{aligned}$$

(Account for the different results in these solutions.)

EXAMPLE 3. By the terms of a will the income at 5% annually of a \$10000 estate goes to a son aged 25 for 10 years or so long as he lives during the 10 years, after which the estate is bequeathed to a charitable institution. Find the value of each legacy.

SOLUTION. The value of the son's legacy is given by

$$\begin{aligned} V &= 500\, a_{25:\overline{10|}} \\ &= 500\,\frac{N_{26} - N_{36}}{D_{25}} \\ &= 500(7.4037) \\ &= \$3702 \end{aligned}$$

By the method of solution 1, Example 2, the value of the remainder is given by

$$\begin{aligned} V &= 10000\, A_{25:\overline{10|}} \\ &= 10000\,\frac{M_{25} - M_{35} + D_{35}}{D_{25}} \\ &= 10000(.626690) \\ &= \$6267 \end{aligned}$$

By the method of solution 2, Example 2, the value of the remainder is given by

$$\begin{aligned} V &= 10000 - 3701.85 \\ &= \$6298 \end{aligned}$$

Example 4. The income at 5% annually of a \$10000 estate is bequeathed to two persons aged 28 and 32 during their joint lives, after which the estate is given to a third party. Find the values of the life estate and the remainder.

Solution. The value of the life estate is given by

$$\begin{aligned} V &= 500\, a_{28:32} \\ &= 500(12.956) \\ &= \$6478 \end{aligned}$$

By the method of solution 1, Example 2, the value of the remainder is given by

$$\begin{aligned} V &= 10000\, A_{28:32} \\ &= 10000(.33542) \\ &= \$3354 \end{aligned}$$

By the method of solution 2, Example 2, the value of the remainder is given by

$$\begin{aligned} V &= 10000 - 6478 \\ &= \$3522 \end{aligned}$$

Example 5. The income at 5% annually of a \$10000 estate is bequeathed to (x) aged 28 and (y) aged 32 equally as long as they both live and all to the survivor as long as he lives. Find the value of the bequest.

Solution. The value of the bequest is given by

$$\begin{aligned} V &= 500(a_{28} + a_{32} - a_{28:32}) \\ &= 500(15.292 + 14.857 - 12.956) \\ &= \$8597 \end{aligned}$$

Example 6. Find the value of each interest in the estate in Example 5.

Solution. The value of the interest of (x) is given by

$$\begin{aligned} V &= 500(a_{28} - \tfrac{1}{2}\, a_{28:32}) \\ &= \$4407 \end{aligned}$$

The value of the interest of (y) is given by

$$\begin{aligned} V &= 500(a_{32} - \tfrac{1}{2}\, a_{28:32}) \\ &= \$4190 \end{aligned}$$

Example 7. A son aged 30 is bequeathed \$500 at the end of each year he lives after the death of his mother, aged 53. Find the value of his legacy.

Solution. The value of his legacy is given by

$$\begin{aligned} V &= 500(a_{30} - a_{30:53}) \\ &= 500(15.08425 - 10.03887) \\ &= \$2522.69 \end{aligned}$$

Exercise 1. If the mother is given the income during her life, find the value of the son's legacy by the method of solution 2, Example 2.

Example 8. The income at 5% annually of a \$10000 estate goes to a hospital until one of two sons aged 28 and 32 dies. Then the income goes to the survivor during his life. Find the value of the survivor's interest.

SOLUTION. The value of this legacy is given by

$$\begin{aligned} V &= 500(a_{28} + a_{32} - 2\,a_{28:32}) \\ &= 500(4.2375) \\ &= \$2119 \end{aligned}$$

EXAMPLE 9. The estate in Example 5 is given to (z) at the death of the survivor of (x) and (y). Find the value of the legacy to (z).

SOLUTION. The value of this survivorship insurance is given by

$$\begin{aligned} V &= 10000(A_{28} + A_{32} - A_{28:32}) \\ &= 10000(.22419 + .24492 - .33542) \\ &= \$1337 \end{aligned}$$

EXERCISE 2. Find the value of this legacy by deducting the value of the bequest to (x) and (y) from 10000.

The above solutions of simple examples in valuing life estates and remainders are based on life annuities and life insurances involving one or two lives. Similar problems involving one, two, three, or more lives can be solved readily by use of the same methods and tables giving the values of commutation symbols for two, three, or more lives of equal ages. For valuing more complex estates additional formulas are needed for the values of contingent annuities and insurances.* In valuing estates in a given state the law of the state pertaining to such valuation should be consulted.

EXERCISE

Solve each of the above examples by use of $3\frac{1}{2}\%$ tables.

128. Inheritance taxes. The determination of inheritance taxes affords one of the important applications of the evaluation of estates. The tax laws of each state which levies inheritance taxes specify the rates and the exemptions for various classes of successors to a given estate. By means of these exemptions and rates the amount of inheritance taxes that must be paid by a given successor can be easily computed from the value of the inheritance. Under the Ohio Law, for example, there are four classes of successors, the first of which consists of a wife or a minor child. For this class the exemption is \$5000 and the rates are as follows:

* See Institute of Actuaries Textbook, Part II, for a full treatment of annuities and insurances.

1% on \$25000 or part thereof over the exemption; 2% on next \$75000 or part thereof; 3% on next \$100000 or part thereof; 4% on the balance. According to this law a widow or a minor child who inherits property whose total value is \$250000 would pay, $25000(.01) + 75000(.02) + 100000(.03) + 45000(.04) = \$6550$ in inheritance taxes.

TABLES

100–115 **TABLE I**

PROPORTIONAL PARTS

	434	433	432	431	430	429	428	427	426	425	424	423
1	43.4	43.3	43.2	43.1	43.0	42.9	42.8	42.7	42.6	42.5	42.4	42.3
2	86.8	86.6	86.4	86.2	86.0	85.8	85.6	85.4	85.2	85.0	84.8	84.6
3	130.2	129.9	129.6	129.3	129.0	128.7	128.4	128.1	127.8	127.5	127.2	126.9
4	173.6	173.2	172.8	172.4	172.0	171.6	171.2	170.8	170.4	170.0	169.6	169.2
5	217.0	216.5	216.0	215.5	215.0	214.5	214.0	213.5	213.0	212.5	212.0	211.5
6	260.4	259.8	259.2	258.6	258.0	257.4	256.8	256.2	255.6	255.0	254.4	253.8
7	303.8	303.1	302.4	301.7	301.0	300.3	299.6	298.9	298.2	297.5	296.8	296.1
8	347.2	346.4	345.6	344.8	344.0	343.2	342.4	341.6	340.8	340.0	339.2	338.4
9	390.6	389.7	388.8	387.9	387.0	386.1	385.2	384.3	383.4	382.5	381.6	380.7

	422	421	420	419	418	417	416	415	414	413	412	411
1	42.2	42.1	42.0	41.9	41.8	41.7	41.6	41.5	41.4	41.3	41.2	41.1
2	84.4	84.2	84.0	83.8	83.6	83.4	83.2	83.0	82.8	82.6	82.4	82.2
3	126.6	126.3	126.0	125.7	125.4	125.1	124.8	124.5	124.2	123.9	123.6	123.3
4	168.8	168.4	168.0	167.6	167.2	166.8	166.4	166.0	165.6	165.2	164.8	164.4
5	211.0	210.5	210.0	209.5	209.0	208.5	208.0	207.5	207.0	206.5	206.0	205.5
6	253.2	252.6	252.0	251.4	250.8	250.2	249.6	249.0	248.4	247.8	247.2	246.6
7	295.4	294.7	294.0	293.3	292.6	291.9	291.2	290.5	289.8	289.1	288.4	287.7
8	337.6	336.8	336.0	335.2	334.4	333.6	332.8	332.0	331.2	330.4	329.6	328.8
9	379.8	378.9	378.0	377.1	376.2	375.3	374.4	373.5	372.6	371.7	370.8	369.9

	410	409	408	407	406	405	404	403	402	401	400	399
1	41.0	40.9	40.8	40.7	40.6	40.5	40.4	40.3	40.2	40.1	40.0	39.9
2	82.0	81.8	81.6	81.4	81.2	81.0	80.8	80.6	80.4	80.2	80.0	79.8
3	123.0	122.7	122.4	122.1	121.8	121.5	121.2	120.9	120.6	120.3	120.0	119.7
4	164.0	163.6	163.2	162.8	162.4	162.0	161.6	161.2	160.8	160.4	160.0	159.6
5	205.0	204.5	204.0	203.5	203.0	202.5	202.0	201.5	201.0	200.5	200.0	199.5
6	246.0	245.4	244.8	244.2	243.6	243.0	242.4	241.8	241.2	240.6	240.0	239.4
7	287.0	286.3	285.6	284.9	284.2	283.5	282.8	282.1	281.4	280.7	280.0	279.3
8	328.0	327.2	326.4	325.6	324.8	324.0	323.2	322.4	321.6	320.8	320.0	319.2
9	369.0	368.1	367.2	366.3	365.4	364.5	363.6	362.7	361.8	360.9	360.0	359.1

	398	397	396	395	394	393	392	391	390	389	388	387
1	39.8	39.7	39.6	39.5	39.4	39.3	39.2	39.1	39.0	38.9	38.8	38.7
2	79.6	79.4	79.2	79.0	78.8	78.6	78.4	78.2	78.0	77.8	77.6	77.4
3	119.4	119.1	118.8	118.5	118.2	117.9	117.6	117.3	117.0	116.7	116.4	116.1
4	159.2	158.8	158.4	158.0	157.6	157.2	156.8	156.4	156.0	155.6	155.2	154.8
5	199.0	198.5	198.0	197.5	197.0	196.5	196.0	195.5	195.0	194.5	194.0	193.5
6	238.8	238.2	237.6	237.0	236.4	235.8	235.2	234.6	234.0	233.4	232.8	232.2
7	278.6	277.9	277.2	276.5	275.8	275.1	274.4	273.7	273.0	272.3	271.6	270.9
8	318.4	317.6	316.8	316.0	315.2	314.4	313.6	312.8	312.0	311.2	310.4	309.6
9	358.2	357.3	356.4	355.5	354.6	353.7	352.8	351.9	351.0	350.1	349.2	348.3

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
100	00 0000	0434	0868	1301	1734	2166	2598	3029	3461	3891
01	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174
02	8600	9026	9451	9876	*0300	*0724	*1147	*1570	*1993	*2415
03	01 2837	3259	3680	4100	4521	4940	5360	5779	6197	6616
04	7033	7451	7868	8284	8700	9116	9532	9947	*0361	*0775
05	02 1189	1603	2016	2428	2841	3252	3664	4075	4486	4896
06	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978
07	9384	9789	*0195	*0600	*1004	*1408	*1812	*2216	*2619	*3021
08	03 3424	3826	4227	4628	5029	5430	5830	6230	6629	7028
09	7426	7825	8223	8620	9017	9414	9811	*0207	*0602	*0998
110	04 1393	1787	2182	2576	2969	3362	3755	4148	4540	4932
11	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830
12	9218	9606	9993	*0380	*0766	*1153	*1538	*1924	*2309	*2694
13	05 3078	3463	3846	4230	4613	4996	5378	5760	6142	6524
14	6905	7286	7666	8046	8426	8805	9185	9563	9942	*0320
115	06 0698	1075	1452	1829	2206	2582	2958	3333	3709	4083

	386	385	384
1	38.6	38.5	38.4
2	77.2	77.0	76.8
3	115.8	115.5	115.2
4	154.4	154.0	153.6
5	193.0	192.5	192.0
6	231.6	231.0	230.4
7	270.2	269.5	268.8
8	308.8	308.0	307.2
9	347.4	346.5	345.6

	383	382	381
1	38.3	38.2	38.1
2	76.6	76.4	76.2
3	114.9	114.6	114.3
4	153.2	152.8	152.4
5	191.5	191.0	190.5
6	229.8	229.2	228.6
7	268.1	267.4	266.7
8	306.4	305.6	304.8
9	344.7	343.8	342.9

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TABLE I

PROPORTIONAL PARTS

	380	379	378	377	376	375	374	373	372	371	370	369
1	38.0	37.9	37.8	37.7	37.6	37.5	37.4	37.3	37.2	37.1	37.0	36.9
2	76.0	75.8	75.6	75.4	75.2	75.0	74.8	74.6	74.4	74.2	74.0	73.8
3	114.0	113.7	113.4	113.1	112.8	112.5	112.2	111.9	111.6	111.3	111.0	110.7
4	152.0	151.6	151.2	150.8	150.4	150.0	149.6	149.2	148.8	148.4	148.0	147.6
5	190.0	189.5	189.0	188.5	188.0	187.5	187.0	186.5	186.0	185.5	185.0	184.5
6	228.0	227.4	226.8	226.2	225.6	225.0	224.4	223.8	223.2	222.6	222.0	221.4
7	266.0	265.3	264.6	263.9	263.2	262.5	261.8	261.1	260.4	259.7	259.0	258.3
8	304.0	303.2	302.4	301.6	300.8	300.0	299.2	298.4	297.6	296.8	296.0	295.2
9	342.0	341.1	340.2	339.3	338.4	337.5	336.6	335.7	334.8	333.9	333.0	332.1

	368	367	366	365	364	363	362	361	360	359	358	357
1	36.8	36.7	36.6	36.5	36.4	36.3	36.2	36.1	36.0	35.9	35.8	35.7
2	73.6	73.4	73.2	73.0	72.8	72.6	72.4	72.2	72.0	71.8	71.6	71.4
3	110.4	110.1	109.8	109.5	109.2	108.9	108.6	108.3	108.0	107.7	107.4	107.1
4	147.2	146.8	146.4	146.0	145.6	145.2	144.8	144.4	144.0	143.6	143.2	142.8
5	184.0	183.5	183.0	182.5	182.0	181.5	181.0	180.5	180.0	179.5	179.0	178.5
6	220.8	220.2	219.6	219.0	218.4	217.8	217.2	216.6	216.0	215.4	214.8	214.2
7	257.6	256.9	256.2	255.7	254.8	254.1	253.4	252.7	252.0	251.3	250.6	249.9
8	294.4	293.6	292.8	292.0	291.2	290.4	289.6	288.8	288.0	287.2	286.4	285.6
9	331.2	330.3	329.4	328.5	327.6	326.7	325.8	324.9	324.0	323.1	322.2	321.3

	356	355	354	353	352	351	350	349	348	347	346	345
1	35.6	35.5	35.4	35.3	35.2	35.1	35.0	34.9	34.8	34.7	34.6	34.5
2	71.2	71.0	70.8	70.6	70.4	70.2	70.0	69.8	69.6	69.4	69.2	69.0
3	106.8	106.5	106.2	105.9	105.6	105.3	105.0	104.7	104.4	104.1	103.8	103.5
4	142.4	142.0	141.6	141.2	140.8	140.4	140.0	139.6	139.2	138.8	138.4	138.0
5	178.0	177.5	177.0	176.5	176.0	175.5	175.0	174.5	174.0	173.5	173.0	172.5
6	213.6	213.0	212.4	211.8	211.2	210.6	210.0	209.4	208.8	208.2	207.6	207.0
7	249.2	248.5	247.8	247.1	246.4	245.7	245.0	244.3	243.6	242.9	242.2	241.5
8	284.8	284.0	283.2	282.4	281.6	280.8	280.0	279.2	278.4	277.6	276.8	276.0
9	320.4	319.5	318.6	317.7	316.8	315.9	315.0	314.1	313.2	312.3	311.4	310.5

	344	343	342	341	340	339	338	337	336	335	334
1	34.4	34.3	34.2	34.1	34.0	33.9	33.8	33.7	33.6	33.5	33.4
2	68.8	68.6	68.4	68.2	68.0	67.8	67.6	67.4	67.2	67.0	66.8
3	103.2	102.9	102.6	102.3	102.0	101.7	101.4	101.1	100.8	100.5	100.2
4	137.6	137.2	136.8	136.4	136.0	135.6	135.2	134.8	134.4	134.0	133.6
5	172.0	171.5	171.0	170.5	170.0	169.5	169.0	168.5	168.0	167.5	167.0
6	206.4	205.8	205.2	204.6	204.0	203.4	202.8	202.0	201.6	201.0	200.4
7	240.8	240.1	239.4	238.7	238.0	237.3	236.6	235.9	235.2	234.5	233.8
8	275.2	274.4	273.6	272.8	272.0	271.2	270.4	269.6	268.8	268.0	267.2
9	309.6	308.7	307.8	306.9	306.0	305.1	304.2	303.3	302.4	301.5	300.6

	333	332
1	33.3	33.2
2	66.6	66.4
3	99.9	99.6
4	133.2	132.8
5	166.5	166.0
6	199.8	199.2
7	233.1	232.4
8	266.4	265.6
9	299.7	298.8

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
115	06 0698	1075	1452	1829	2206	2582	2958	3333	3709	4083
16	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815
17	8186	8557	8928	9298	9668	*0038	*0407	*0776	*1145	*1514
18	07 1882	2250	2617	2985	3352	3718	4085	4451	4816	5182
19	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819
120	07 9181	9543	9904	*0266	*0626	*0987	*1347	*1707	*2067	*2426
21	08 2785	3144	3503	3861	4219	4576	4934	5291	5647	6004
22	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552
23	9905	*0258	*0611	*0963	*1315	*1667	*2018	*2370	*2721	*3071
24	09 3422	3772	4122	4471	4820	5169	5518	5866	6215	6562
25	6910	7257	7604	7951	8298	8644	8990	9335	9681	*0026
26	10 0371	0715	1059	1403	1747	2091	2434	2777	3119	3462
27	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871
28	7210	7549	7888	8227	8565	8903	9241	9579	9916	*0253
29	11 0590	0926	1263	1599	1934	2270	2605	2940	3275	3609
130	11 3943	4277	4611	4944	5278	5611	5943	6276	6608	6940

130–150 **TABLE I**

	Proportional Parts				Proportional Parts				Proportional Parts				Proportional Parts		
	331	**330**	**329**		**328**	**327**	**326**		**325**	**324**	**323**		**322**	**321**	**320**
1	33.1	33.0	32.9	**1**	32.8	32.7	32.6	**1**	32.5	32.4	32.3	**1**	32.2	32.1	32.0
2	66.2	66.0	65.8	**2**	65.6	65.4	65.2	**2**	65.0	64.8	64.6	**2**	64.4	64.2	64.0
3	99.3	99.0	98.7	**3**	98.4	98.1	97.8	**3**	97.5	97.2	96.9	**3**	96.6	96.3	96.0
4	132.4	132.0	131.6	**4**	131.2	130.8	130.4	**4**	130.0	129.6	129.2	**4**	128.8	128.4	128.0
5	165.5	165.0	164.5	**5**	164.0	163.5	163.0	**5**	162.5	162.0	161.5	**5**	161.0	160.5	160.0
6	198.6	198.0	197.4	**6**	196.8	196.2	195.6	**6**	195.0	194.4	193.8	**6**	193.2	192.6	192.0
7	231.7	231.0	230.3	**7**	229.6	228.9	228.2	**7**	227.5	226.8	226.1	**7**	225.4	224.7	224.0
8	264.8	264.0	263.2	**8**	262.4	261.6	260.8	**8**	260.0	259.2	258.4	**8**	257.6	256.8	256.0
9	297.9	297.0	296.1	**9**	295.2	294.3	293.4	**9**	292.5	291.6	290.7	**9**	289.8	288.9	288.0
	319	**318**	**317**		**316**	**315**	**314**		**313**	**312**	**311**		**310**	**309**	**308**
1	31.9	31.8	31.7	**1**	31.6	31.5	31.4	**1**	31.3	31.2	31.1	**1**	31.0	30.9	30.8
2	63.8	63.6	63.4	**2**	63.2	63.0	62.8	**2**	62.6	62.4	62.2	**2**	62.0	61.8	61.6
3	95.7	95.4	95.1	**3**	94.8	94.5	94.2	**3**	93.9	93.6	93.3	**3**	93.0	92.7	92.4
4	127.6	127.2	126.8	**4**	126.4	126.0	125.6	**4**	125.2	124.8	124.4	**4**	124.0	123.6	123.2
5	159.5	159.0	158.5	**5**	158.0	157.5	157.0	**5**	156.5	156.0	155.5	**5**	155.0	154.5	154.0
6	191.4	190.8	190.2	**6**	189.6	189.0	188.4	**6**	187.8	187.2	186.6	**6**	186.0	185.4	184.8
7	223.3	222.6	221.9	**7**	221.2	220.5	219.8	**7**	219.1	218.4	217.7	**7**	217.0	216.3	215.6
8	255.2	254.4	253.6	**8**	252.8	252.0	251.2	**8**	250.4	249.6	248.8	**8**	248.0	247.2	246.4
9	287.1	286.2	285.3	**9**	284.4	283.5	282.6	**9**	281.7	280.8	279.9	**9**	279.0	278.1	277.2
	307	**306**	**305**		**304**	**303**	**302**		**301**	**300**	**299**		**298**	**297**	**296**
1	30.7	30.6	30.5	**1**	30.4	30.3	30.2	**1**	30.1	30.0	29.9	**1**	29.8	29.7	29.6
2	61.4	61.2	61.0	**2**	60.8	60.6	60.4	**2**	60.2	60.0	59.8	**2**	59.6	59.4	59.2
3	92.1	91.8	91.5	**3**	91.2	90.9	90.6	**3**	90.3	90.0	89.7	**3**	89.4	89.1	88.8
4	122.8	122.4	122.0	**4**	121.6	121.2	120.8	**4**	120.4	120.0	119.6	**4**	119.2	118.8	118.4
5	153.5	153.0	152.5	**5**	152.0	151.5	151.0	**5**	150.5	150.0	149.5	**5**	149.0	148.5	148.0
6	184.2	183.6	183.0	**6**	182.4	181.8	181.2	**6**	180.6	180.0	179.4	**6**	178.8	178.2	177.6
7	214.9	214.2	213.5	**7**	212.8	212.1	211.4	**7**	210.7	210.0	209.3	**7**	208.6	207.9	207.2
8	245.6	244.8	244.0	**8**	243.2	242.4	241.6	**8**	240.8	240.0	239.2	**8**	238.4	237.6	236.8
9	276.3	275.4	274.5	**9**	273.6	272.7	271.8	**9**	270.9	270.0	269.1	**9**	268.2	267.3	266.4

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
130	11 3943	4277	4611	4944	5278	5611	5943	6276	6608	6940
31	7271	7603	7934	8265	8595	8926	9256	9586	9915	*0245
32	12 0574	0903	1231	1560	1888	2216	2544	2871	3198	3525
33	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781
34	7105	7429	7753	8076	8399	8722	9045	9368	9690	*0012
35	13 0334	0655	0977	1298	1619	1939	2260	2580	2900	3219
36	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403
37	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564
38	9879	*0194	*0508	*0822	*1136	*1450	*1763	*2076	*2389	*2702
39	14 3015	3327	3639	3951	4263	4574	4885	5196	5507	5818
140	14 6128	6438	6748	7058	7367	7676	7985	8294	8603	8911
41	9219	9527	9835	*0142	*0449	*0756	*1063	*1370	*1676	*1982
42	15 2288	2594	2900	3205	3510	3815	4120	4424	4728	5032
43	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061
44	8362	8664	8965	9266	9567	9868	*0168	*0469	*0769	*1068
45	16 1368	1667	1967	2266	2564	2863	3161	3460	3758	4055
46	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022
47	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968
48	17 0262	0555	0848	1141	1434	1726	2019	2311	2603	2895
49	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802
150	6091	6381	6670	6959	7248	7536	7825	8113	8401	8689

	Proportional Parts		
	295	**294**	**293**
1	29.5	29.4	29.3
2	59.0	58.8	58.6
3	88.5	88.2	87.9
4	118.0	117.6	117.2
5	147.5	147.0	146.5
6	177.0	176.4	175.8
7	206.5	205.8	205.1
8	236.0	235.2	234.4
9	265.5	264.6	263.7
	292	**291**	**290**
1	29.2	29.1	29.0
2	58.4	58.2	58.0
3	87.6	87.3	87.0
4	116.8	116.4	116.0
5	146.0	145.5	145.0
6	175.2	174.6	174.0
7	204.4	203.7	203.0
8	233.6	232.8	232.0
9	262.8	261.9	261.0
	289	**288**	**287**
1	28.9	28.8	28.7
2	57.8	57.6	57.4
3	86.7	86.4	86.1
4	115.6	115.2	114.8
5	144.5	144.0	143.5
6	173.4	172.8	172.2
7	202.3	201.6	200.9
8	231.2	230.4	229.6
9	260.1	259.2	258.3

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TABLE I

Proportional Parts

	290	289	288		287	286	285		284	283	282		281	280	279
1	29.0	28.9	28.8	1	28.7	28.6	28.5	1	28.4	28.3	28.2	1	28.1	28.0	27.9
2	58.0	57.8	57.6	2	57.4	57.2	57.0	2	56.8	56.6	56.4	2	56.2	56.0	55.8
3	87.0	86.7	86.4	3	86.1	85.8	85.5	3	85.2	84.9	84.6	3	84.3	84.0	83.7
4	116.0	115.6	115.2	4	114.8	114.4	114.0	4	113.6	113.2	112.8	4	112.4	112.0	111.6
5	145.0	144.5	144.0	5	143.5	143.0	142.5	5	142.0	141.5	141.0	5	140.5	140.0	139.5
6	174.0	173.4	172.8	6	172.2	171.6	171.0	6	170.4	169.8	169.2	6	168.6	168.0	167.4
7	203.0	202.3	201.6	7	200.9	200.2	199.5	7	198.8	198.1	197.4	7	196.7	196.0	195.3
8	232.0	231.2	230.4	8	229.6	228.8	228.0	8	227.2	226.4	225.6	8	224.8	224.0	223.2
9	261.0	260.1	259.2	9	258.3	257.4	256.5	9	255.6	254.7	253.8	9	252.9	252.0	251.1

	278	277	276		275	274	273		272	271	270		269	268	267
1	27.8	27.7	27.6	1	27.5	27.4	27.3	1	27.2	27.1	27.0	1	26.9	26.8	26.7
2	55.6	55.4	55.2	2	55.0	54.8	54.6	2	54.4	54.2	54.0	2	53.8	53.6	53.4
3	83.4	83.1	82.8	3	82.5	82.2	81.9	3	81.6	81.3	81.0	3	80.7	80.4	80.1
4	111.2	110.8	110.4	4	110.0	109.6	109.2	4	108.8	108.4	108.0	4	107.6	107.2	106.8
5	139.0	138.5	138.0	5	137.5	137.0	136.5	5	136.0	135.5	135.0	5	134.5	134.0	133.5
6	166.8	166.2	165.6	6	165.0	164.4	163.8	6	163.2	162.6	162.0	6	161.4	160.8	160.2
7	194.6	193.9	193.2	7	192.5	191.8	191.1	7	190.4	189.7	189.0	7	188.3	187.6	186.9
8	222.4	221.6	220.8	8	220.0	219.2	218.4	8	217.6	216.8	216.0	8	215.2	214.4	213.6
9	250.2	249.3	248.4	9	247.5	246.6	245.7	9	244.8	243.9	243.0	9	242.1	241.2	240.3

	266	265	264		263	262	261		260	259	258		257	256	255
1	26.6	26.5	26.4	1	26.3	26.2	26.1	1	26.0	25.9	25.8	1	25.7	25.6	25.5
2	53.2	53.0	52.8	2	52.6	52.4	52.2	2	52.0	51.8	51.6	2	51.4	51.2	51.0
3	79.8	79.5	79.2	3	78.9	78.6	78.3	3	78.0	77.7	77.4	3	77.1	76.8	76.5
4	106.4	106.0	105.6	4	105.2	104.8	104.4	4	104.0	103.6	103.2	4	102.8	102.4	102.0
5	133.0	132.5	132.0	5	131.5	131.0	130.5	5	130.0	129.5	129.0	5	128.5	128.0	127.5
6	159.6	159.0	158.4	6	157.8	157.2	156.6	6	156.0	155.4	154.8	6	154.2	153.6	153.0
7	186.2	185.5	184.8	7	184.1	183.4	182.7	7	182.0	181.3	180.6	7	179.9	179.2	178.5
8	212.8	212.0	211.2	8	210.4	209.6	208.8	8	208.0	207.2	206.4	8	205.6	204.8	204.0
9	239.4	238.5	237.6	9	236.7	235.8	234.9	9	234.0	233.1	232.2	9	231.3	230.4	229.5

	254
1	25.4
2	50.8
3	76.2
4	101.6
5	127.0
6	152.4
7	177.8
8	203.2
9	228.6

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
150	17 6091	6381	6670	6959	7248	7536	7825	8113	8401	8689
51	8977	9264	9552	9839	*0126	*0413	*0699	*0986	*1272	*1558
52	18 1844	2129	2415	2700	2985	3270	3555	3839	4123	4407
53	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239
54	7521	7803	8084	8366	8647	8928	9209	9490	9771	*0051
55	19 0332	0612	0892	1171	1451	1730	2010	2289	2567	2846
56	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623
57	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382
58	8657	8932	9206	9481	9755	*0029	*0303	*0577	*0850	*1124
59	20 1397	1670	1943	2216	2488	2761	3033	3305	3577	3848
160	20 4120	4391	4663	4934	5204	5475	5746	6016	6286	6556
61	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247
62	9515	9783	*0051	*0319	*0586	*0853	*1121	*1388	*1654	*1921
63	21 2188	2454	2720	2986	3252	3518	3783	4049	4314	4579
64	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221
65	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846
66	22 0108	0370	0631	0892	1153	1414	1675	1936	2196	2456
67	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051
68	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630
69	7887	8144	8400	8657	8913	9170	9426	9682	9938	*0193
170	23 0449	0704	0960	1215	1470	1724	1979	2234	2488	2742

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TABLE I

PROPORTIONAL PARTS

	255	254	253		252	251	250		249	248	247
1	25.5	25.4	25.3	1	25.2	25.1	25.0	1	24.9	24.8	24.7
2	51.0	50.8	50.6	2	50.4	50.2	50.0	2	49.8	49.6	49.4
3	76.5	76.2	75.9	3	75.6	75.3	75.0	3	74.7	74.4	74.1
4	102.0	101.6	101.2	4	100.8	100.4	100.0	4	99.6	99.2	98.8
5	127.5	127.0	126.5	5	126.0	125.5	125.0	5	124.5	124.0	123.5
6	153.0	152.4	151.8	6	151.2	150.6	150.0	6	149.4	148.8	148.2
7	178.5	177.8	177.1	7	176.4	175.7	175.0	7	174.3	173.6	172.9
8	204.0	203.2	202.4	8	201.6	200.8	200.0	8	199.2	198.4	197.6
9	229.5	228.6	227.7	9	226.8	225.9	225.0	9	224.1	223.2	222.3
	243	**242**	**241**		**240**	**239**	**238**		**237**	**236**	**235**
1	24.3	24.2	24.1	1	24.0	23.9	23.8	1	23.7	23.6	23.5
2	48.6	48.4	48.2	2	48.0	47.8	47.6	2	47.4	47.2	47.0
3	72.9	72.6	72.3	3	72.0	71.7	71.4	3	71.1	70.8	70.5
4	97.2	96.8	96.4	4	96.0	95.6	95.2	4	94.8	94.4	94.0
5	121.5	121.0	120.5	5	120.0	119.5	119.0	5	118.5	118.0	117.5
6	145.8	145.2	144.6	6	144.0	143.4	142.8	6	142.2	141.6	141.0
7	170.1	169.4	168.7	7	168.0	167.3	166.6	7	165.9	165.2	164.5
8	194.4	193.6	192.8	8	192.0	191.2	190.4	8	189.6	188.8	188.0
9	218.7	217.8	216.9	9	216.0	215.1	214.2	9	213.3	212.4	211.5

PROPORTIONAL PARTS

	246	245	244
1	24.6	24.5	24.4
2	49.2	49.0	48.8
3	73.8	73.5	73.2
4	98.4	98.0	97.6
5	123.0	122.5	122.0
6	147.6	147.0	146.4
7	172.2	171.5	170.8
8	196.8	196.0	195.2
9	221.4	220.5	219.6
	234	**233**	**232**
1	23.4	23.3	23.2
2	46.8	46.6	46.4
3	70.2	69.9	69.6
4	93.6	93.2	92.8
5	117.0	116.5	116.0
6	140.4	139.8	139.2
7	163.8	163.1	162.4
8	187.2	186.4	185.6
9	210.6	209.7	208.8
	231	**230**	**229**
1	23.1	23.0	22.9
2	46.2	46.0	45.8
3	69.3	69.0	68.7
4	92.4	92.0	91.6
5	115.5	115.0	114.5
6	138.6	138.0	137.4
7	161.7	161.0	160.3
8	184.8	184.0	183.2
9	207.9	207.0	206.1
	228	**227**	**226**
1	22.8	22.7	22.6
2	45.6	45.4	45.2
3	68.4	68.1	67.8
4	91.2	90.8	90.4
5	114.0	113.5	113.0
6	136.8	136.2	135.6
7	159.6	158.9	158.2
8	182.4	181.6	180.8
9	205.2	204.3	203.4
	225	**224**	**223**
1	22.5	22.4	22.3
2	45.0	44.8	44.6
3	67.5	67.2	66.9
4	90.0	89.6	89.2
5	112.5	112.0	111.5
6	135.0	134.4	133.8
7	157.5	156.8	156.1
8	180.0	179.2	178.4
9	202.5	201.6	200.7
	222	**221**	**220**
1	22.2	22.1	22.0
2	44.4	44.2	44.0
3	66.6	66.3	66.0
4	88.8	88.4	88.0
5	111.0	110.5	110.0
6	133.2	132.6	132.0
7	155.4	154.7	154.0
8	177.6	176.8	176.0
9	199.8	198.9	198.0

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
170	23 0449	0704	0960	1215	1470	1724	1979	2234	2488	2742
71	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276
72	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795
73	8046	8297	8548	8799	9049	9299	9550	9800	*0050	*0300
74	24 0549	0799	1048	1297	1546	1795	2044	2293	2541	2790
75	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266
76	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728
77	7973	8219	8464	8709	8954	9198	9443	9687	9932	*0176
78	25 0420	0664	0908	1151	1395	1638	1881	2125	2368	2610
79	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031
180	25 5273	5514	5755	5996	6237	6477	6718	6958	7198	7439
81	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833
82	26 0071	0310	0548	0787	1025	1263	1501	1739	1976	2214
83	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582
84	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937
85	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279
86	9513	9746	9980	*0213	*0446	*0679	*0912	*1144	*1377	*1609
87	27 1842	2074	2306	2538	2770	3001	3233	3464	3696	3927
88	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232
89	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525
190	27 8754	8982	9211	9439	9667	9895	*0123	*0351	*0578	*0806
91	28 1033	1261	1488	1715	1942	2169	2396	2622	2849	3075
92	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332
93	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578
94	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812
95	29 0035	0257	0480	0702	0925	1147	1369	1591	1813	2034
96	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246
97	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446
98	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635
99	8853	9071	9289	9507	9725	9943	*0161	*0378	*0595	*0813
200	30 1030	1247	1464	1681	1898	2114	2331	2547	2764	2980

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TABLE I

PROPORTIONAL PARTS

	219	218	217		216	215	214		213	212	211		210	209	208
1	21.9	21.8	21.7	1	21.6	21.5	21.4	1	21.3	21.2	21.1	1	21.0	20.9	20.8
2	43.8	43.6	43.4	2	43.2	43.0	42.8	2	42.6	42.4	42.2	2	42.0	41.8	41.6
3	65.7	65.4	65.1	3	64.8	64.5	64.2	3	63.9	63.6	63.3	3	63.0	62.7	62.4
4	87.6	87.2	86.8	4	86.4	86.0	85.6	4	85.2	84.8	84.4	4	84.0	83.6	83.2
5	109.5	109.0	108.5	5	108.0	107.5	107.0	5	106.5	106.0	105.5	5	105.0	104.5	104.0
6	131.4	130.8	130.2	6	129.6	129.0	128.4	6	127.8	127.2	126.6	6	126.0	125.4	124.8
7	153.3	152.6	151.9	7	151.2	150.5	149.8	7	149.1	148.4	147.7	7	147.0	146.3	145.6
8	175.2	174.4	173.6	8	172.8	172.0	171.2	8	170.4	169.6	168.8	8	168.0	167.2	166.4
9	197.1	196.2	195.3	9	194.4	193.5	192.6	9	191.7	190.8	189.9	9	189.0	188.1	187.2

	207	206	205		204	203	202		201	200	199		198	197	196
1	20.7	20.6	20.5	1	20.4	20.3	20.2	1	20.1	20.0	19.9	1	19.8	19.7	19.6
2	41.4	41.2	41.0	2	40.8	40.6	40.4	2	40.2	40.0	39.8	2	39.6	39.4	39.2
3	62.1	61.8	61.5	3	61.2	60.9	60.6	3	60.3	60.0	59.7	3	59.4	59.1	58.8
4	82.8	82.4	82.0	4	81.6	81.2	80.8	4	80.4	80.0	79.6	4	79.2	78.8	78.4
5	103.5	103.0	102.5	5	102.0	101.5	101.0	5	100.5	100.0	99.5	5	99.0	98.5	98.0
6	124.2	123.6	123.0	6	122.4	121.8	121.2	6	120.6	120.0	119.4	6	118.8	118.2	117.6
7	144.9	144.2	143.5	7	142.8	142.1	141.4	7	140.7	140.0	139.3	7	138.6	137.9	137.2
8	165.6	164.8	164.0	8	163.2	162.4	161.6	8	160.8	160.0	159.2	8	158.4	157.6	156.8
9	186.3	185.4	184.5	9	183.6	182.7	181.8	9	180.9	180.0	179.1	9	178.2	177.3	176.4

	195	194	193
1	19.5	19.4	19.3
2	39.0	38.8	38.6
3	58.5	58.2	57.9
4	78.0	77.6	77.2
5	97.5	97.0	96.5
6	117.0	116.4	115.8
7	136.5	135.8	135.1
8	156.0	155.2	154.4
9	175.5	174.6	173.7

	192	191	190
1	19.2	19.1	19.0
2	38.4	38.2	38.0
3	57.6	57.3	57.0
4	76.8	76.4	76.0
5	96.0	95.5	95.0
6	115.2	114.6	114.0
7	134.4	133.7	133.0
8	153.6	152.8	152.0
9	172.8	171.9	171.0

	189
1	18.9
2	37.8
3	56.7
4	75.6
5	94.5
6	113.4
7	132.3
8	151.2
9	170.1

	188
1	18.8
2	37.6
3	56.4
4	75.2
5	94.0
6	112.8
7	131.6
8	150.4
9	169.2

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
200	30 1030	1247	1464	1681	1898	2114	2331	2547	2764	2980
01	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136
02	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282
03	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417
04	9630	9843	*0056	*0268	*0481	*0693	*0906	*1118	*1330	*1542
05	31 1754	1966	2177	2389	2600	2812	3023	3234	3445	3656
06	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760
07	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854
08	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938
09	32 0146	0354	0562	0769	0977	1184	1391	1598	1805	2012
210	32 2219	2426	2633	2839	3046	3252	3458	3665	3871	4077
11	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131
12	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176
13	8380	8583	8787	8991	9194	9398	9601	9805	*0008	*0211
14	33 0414	0617	0819	1022	1225	1427	1630	1832	2034	2236
15	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253
16	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260
17	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257
18	8456	8656	8855	9054	9253	9451	9650	9849	*0047	*0246
19	34 0444	0642	0841	1039	1237	1435	1632	1830	2028	2225
220	34 2423	2620	2817	3014	3212	3409	3606	3802	3999	4196
21	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157
22	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110
23	8305	8500	8694	8889	9083	9278	9472	9666	9860	*0054
24	35 0248	0442	0636	0829	1023	1216	1410	1603	1796	1989
25	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916
26	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834
27	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744
28	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646
29	9835	*0025	*0215	*0404	*0593	*0783	*0972	*1161	*1350	*1539
230	36 1728	1917	2105	2294	2482	2671	2859	3048	3236	3424

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TABLE I

	Proportional Parts				Proportional Parts				Proportional Parts		
	189	188	187		186	185	184		183	182	181
1	18.9	18.8	18.7	1	18.6	18.5	18.4	1	18.3	18.2	18.1
2	37.8	37.6	37.4	2	37.2	37.0	36.8	2	36.6	36.4	36.2
3	56.7	56.4	56.1	3	55.8	55.5	55.2	3	54.9	54.6	54.3
4	75.6	75.2	74.8	4	74.4	74.0	73.6	4	73.2	72.8	72.4
5	94.5	94.0	93.5	5	93.0	92.5	92.0	5	91.5	91.0	90.5
6	113.4	112.8	112.2	6	111.6	111.0	110.4	6	109.8	109.2	108.6
7	132.3	131.6	130.9	7	130.2	129.5	128.8	7	128.1	127.4	126.7
8	151.2	150.4	149.6	8	148.8	148.0	147.2	8	146.4	145.6	144.8
9	170.1	169.2	168.3	9	167.4	166.5	165.6	9	164.7	163.8	162.9

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
230	36 1728	1917	2105	2294	2482	2671	2859	3048	3236	3424
31	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301
32	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169
33	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030
34	9216	9401	9587	9772	9958	*0143	*0328	*0513	*0698	*0883
35	37 1068	1253	1437	1622	1806	1991	2175	2360	2544	2728
36	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565
37	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394
38	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216
39	8398	8580	8761	8943	9124	9306	9487	9668	9849	*0030
240	38 0211	0392	0573	0754	0934	1115	1296	1476	1656	1837
41	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636
42	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428
43	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212
44	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989
45	9166	9343	9520	9698	9875	*0051	*0228	*0405	*0582	*0759
46	39 0935	1112	1288	1464	1641	1817	1993	2169	2345	2521
47	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277
48	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025
49	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766
250	39 7940	8114	8287	8461	8634	8808	8981	9154	9328	9501
51	9674	9847	*0020	*0192	*0365	*0538	*0711	*0883	*1056	*1228
52	40 1401	1573	1745	1917	2089	2261	2433	2605	2777	2949
53	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663
54	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370
55	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070
56	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764
57	9933	*0102	*0271	*0440	*0609	*0777	*0946	*1114	*1283	*1451
58	41 1620	1788	1956	2124	2293	2461	2629	2796	2964	3132
59	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806
260	41 4973	5140	5307	5474	5641	5808	5974	6141	6308	6474
61	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135
62	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791
63	9956	*0121	*0286	*0451	*0616	*0781	*0945	*1110	*1275	*1439
64	42 1604	1768	1933	2097	2261	2426	2590	2754	2918	3082
65	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718
66	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349
67	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973
68	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591
69	9752	9914	*0075	*0236	*0398	*0559	*0720	*0881	*1042	*1203
270	43 1364	1525	1685	1846	2007	2167	2328	2488	2649	2800

	Proportional Parts		
	180	179	178
1	18.0	17.9	17.8
2	36.0	35.8	35.6
3	54.0	53.7	53.4
4	72.0	71.6	71.2
5	90.0	89.5	89.0
6	108.0	107.4	106.8
7	126.0	125.3	124.6
8	144.0	143.2	142.4
9	162.0	161.1	160.2
	177	176	175
1	17.7	17.6	17.5
2	35.4	35.2	35.0
3	53.1	52.8	52.5
4	70.8	70.4	70.0
5	88.5	88.0	87.5
6	106.2	105.6	105.0
7	123.9	123.2	122.5
8	141.6	140.8	140.0
9	159.3	158.4	157.5
	174	173	172
1	17.4	17.3	17.2
2	34.8	34.6	34.4
3	52.2	51.9	51.6
4	69.6	69.2	68.8
5	87.0	86.5	86.0
6	104.4	103.8	103.2
7	121.8	121.1	120.4
8	139.2	138.4	137.6
9	156.6	155.7	154.8
	171	170	169
1	17.1	17.0	16.9
2	34.2	34.0	33.8
3	51.3	51.0	50.7
4	68.4	68.0	67.6
5	85.5	85.0	84.5
6	102.6	102.0	101.4
7	119.7	119.0	118.3
8	136.8	136.0	135.2
9	153.9	153.0	152.1
	168	167	166
1	16.8	16.7	16.6
2	33.6	33.4	33.2
3	50.4	50.1	49.8
4	67.2	66.8	66.4
5	84.0	83.5	83.0
6	100.8	100.2	99.6
7	117.6	116.9	116.2
8	134.4	133.6	132.8
9	151.2	150.3	149.4
	165	164	163
1	16.5	16.4	16.3
2	33.0	32.8	32.6
3	49.5	49.2	48.9
4	66.0	65.6	65.2
5	82.5	82.0	81.5
6	99.0	98.4	97.8
7	115.5	114.8	114.1
8	132.0	131.2	130.4
9	148.5	147.6	146.7

Proportional Parts

	162	161	160		159	158	157		156	155	154
1	16.2	16.1	16.0	1	15.9	15.8	15.7	1	15.6	15.5	15.4
2	32.4	32.2	32.0	2	31.8	31.6	31.4	2	31.2	31.0	30.8
3	48.6	48.3	48.0	3	47.7	47.4	47.1	3	46.8	46.5	46.2
4	64.8	64.4	64.0	4	63.6	63.2	62.8	4	62.4	62.0	61.6
5	81.0	80.5	80.0	5	79.5	79.0	78.5	5	78.0	77.5	77.0
6	97.2	96.6	96.0	6	95.4	94.8	94.2	6	93.6	93.0	92.4
7	113.4	112.7	112.0	7	111.3	110.6	109.9	7	109.2	108.5	107.8
8	129.6	128.8	128.0	8	127.2	126.4	125.6	8	124.8	124.0	123.2
9	145.8	144.9	144.0	9	143.1	142.2	141.3	9	140.4	139.5	138.6

Proportional Parts

	153	152	151
1	15.3	15.2	15.1
2	30.6	30.4	30.2
3	45.9	45.6	45.3
4	61.2	60.8	60.4
5	76.5	76.0	75.5
6	91.8	91.2	90.6
7	107.1	106.4	105.7
8	122.4	121.6	120.8
9	137.7	136.8	135.9

	150	149	148
1	15.0	14.9	14.8
2	30.0	29.8	29.6
3	45.0	44.7	44.4
4	60.0	59.6	59.2
5	75.0	74.5	74.0
6	90.0	89.4	88.8
7	105.0	104.3	103.6
8	120.0	119.2	118.4
9	135.0	134.1	133.2

	147	146	145
1	14.7	14.6	14.5
2	29.4	29.2	29.0
3	44.1	43.8	43.5
4	58.8	58.4	58.0
5	73.5	73.0	72.5
6	88.2	87.6	87.0
7	102.9	102.2	101.5
8	117.6	116.8	116.0
9	132.3	131.4	130.5

	144	143
1	14.4	14.3
2	28.8	28.6
3	43.2	42.9
4	57.6	57.2
5	72.0	71.5
6	86.4	85.8
7	100.8	100.1
8	115.2	114.4
9	129.6	128.7

	142	141
1	14.2	14.1
2	28.4	28.2
3	42.6	42.3
4	56.8	56.4
5	71.0	70.5
6	85.2	84.6
7	99.4	98.7
8	113.6	112.8
9	127.8	126.9

	140
1	14.0
2	28.0
3	42.0
4	56.0
5	70.0
6	84.0
7	98.0
8	112.0
9	126.0

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
270	43 1364	1525	1685	1846	2007	2167	2328	2488	2649	2809
71	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409
72	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004
73	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592
74	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175
75	9333	9491	9648	9806	9964	*0122	*0279	*0437	*0594	*0752
76	44 0909	1066	1224	1381	1538	1695	1852	2009	2166	2323
77	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889
78	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449
79	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003
280	44 7158	7313	7468	7623	7778	7933	8088	8242	8397	8552
81	8706	8861	9015	9170	9324	9478	9633	9787	9941	*0095
82	45 0249	0403	0557	0711	0865	1018	1172	1326	1479	1633
83	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165
84	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692
85	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214
86	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731
87	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242
88	9392	9543	9694	9845	9995	*0146	*0296	*0447	*0597	*0748
89	46 0898	1048	1198	1348	1499	1649	1799	1948	2098	2248
290	46 2398	2548	2697	2847	2997	3146	3296	3445	3594	3744
91	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234
92	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719
93	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200
94	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675
95	9822	9969	*0116	*0263	*0410	*0557	*0704	*0851	*0998	*1145
96	47 1292	1438	1585	1732	1878	2025	2171	2318	2464	2610
97	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071
98	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526
99	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976
300	47 7121	7266	7411	7555	7700	7844	7989	8133	8278	8422
01	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863
02	48 0007	0151	0294	0438	0582	0725	0869	1012	1156	1299
03	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731
04	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157
05	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579
06	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997
07	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410
08	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818
09	9958	*0099	*0239	*0380	*0520	*0661	*0801	*0941	*1081	*1222
310	49 1362	1502	1642	1782	1922	2062	2201	2341	2481	2621

310–350 **TABLE I**

PROPORTIONAL PARTS

	140	139	138		137	136	135		134	133	132		131	130	129
1	14.0	13.9	13.8	1	13.7	13.6	13.5	1	13.4	13.3	13.2	1	13.1	13.0	12.9
2	28.0	27.8	27.6	2	27.4	27.2	27.0	2	26.8	26.6	26.4	2	26.2	26.0	25.8
3	42.0	41.7	41.4	3	41.1	40.8	40.5	3	40.2	39.9	39.6	3	39.3	39.0	38.7
4	56.0	55.6	55.2	4	54.8	54.4	54.0	4	53.6	53.2	52.8	4	52.4	52.0	51.6
5	70.0	69.5	69.0	5	68.5	68.0	67.5	5	67.0	66.5	66.0	5	65.5	65.0	64.5
6	84.0	83.4	82.8	6	82.2	81.6	81.0	6	80.4	79.8	79.2	6	78.6	78.0	77.4
7	98.0	97.3	96.6	7	95.9	95.2	94.5	7	93.8	93.1	92.4	7	91.7	91.0	90.3
8	112.0	111.2	110.4	8	109.6	108.8	108.0	8	107.2	106.4	105.6	8	104.8	104.0	103.2
9	126.0	125.1	124.2	9	123.3	122.4	121.5	9	120.6	119.7	118.8	9	117.9	117.0	116.1

	128	127
1	12.8	12.7
2	25.6	25.4
3	38.4	38.1
4	51.2	50 8
5	64.0	63.5
6	76.8	76.2
7	89.6	88.9
8	102.4	101.6
9	115.2	114.3

	126	125
1	12.6	12.5
2	25.2	25.0
3	37.8	37.5
4	50.4	50.0
5	63.0	62.5
6	75.6	75.0
7	88.2	87.5
8	100.8	100.0
9	113.4	112.5

	124
1	12.4
2	24.8
3	37.2
4	49.6
5	62.0
6	74.4
7	86.8
8	99.2
9	111.6

	123
1	12.3
2	24.6
3	36.9
4	49.2
5	61.5
6	73.8
7	86.1
8	98.4
9	110.7

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
310	49 1362	1502	1642	1782	1922	2062	2201	2341	2481	2621
11	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015
12	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406
13	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791
14	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173
15	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550
16	9687	9824	9962	*0099	*0236	*0374	*0511	*0648	*0785	*0922
17	50 1059	1196	1333	1470	1607	1744	1880	2017	2154	2291
18	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655
19	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014
320	50 5150	5286	5421	5557	5693	5828	5964	6099	6234	6370
21	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721
22	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068
23	9203	9337	9471	9606	9740	9874	*0009	*0143	*0277	*0411
24	51 0545	0679	0813	0947	1081	1215	1349	1482	1616	1750
25	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084
26	3218	3351	3484	3617	3750	3883	4016	4149	4282	4415
27	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741
28	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064
29	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382
330	51 8514	8646	8777	8909	9040	9171	9303	9434	9566	9697
31	9828	9959	*0090	*0221	*0353	*0484	*0615	*0745	*0876	*1007
32	52 1138	1269	1400	1530	1661	1792	1922	2053	2183	2314
33	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616
34	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915
35	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210
36	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501
37	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788
38	8917	9045	9174	9302	9430	9559	9687	9815	9943	*0072
39	53 0200	0328	0456	0584	0712	0840	0968	1096	1223	1351
340	53 1479	1607	1734	1862	1990	2117	2245	2372	2500	2627
41	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899
42	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167
43	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432
44	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693
45	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951
46	9076	9202	9327	9452	9578	9703	9829	9954	*0079	*0204
47	54 0329	0455	0580	0705	0830	0955	1080	1205	1330	1454
48	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701
49	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944
350	54 4068	4192	4316	4440	4564	4688	4812	4937	5060	5183

350–400 **TABLE I**

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
350	54 4068	4192	4316	4440	4564	4688	4812	4936	5060	5183
51	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419
52	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652
53	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881
54	9003	9126	9249	9371	9494	9616	9739	9861	9984	*0106
55	55 0228	0351	0473	0595	0717	0840	0962	1084	1206	1328
56	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547
57	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762
58	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973
59	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182
360	55 6303	6423	6544	6664	6785	6905	7026	7146	7267	7387
61	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589
62	8709	8829	8949	9068	9188	9308	9428	9548	9667	9787
63	9907	*0026	*0146	*0265	*0385	*0504	*0624	*0743	*0863	*0982
64	56 1101	1221	1340	1459	1578	1698	1817	1936	2055	2174
65	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362
66	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548
67	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730
68	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909
69	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084
370	56 8202	8319	8436	8554	8671	8788	8905	9023	9140	9257
71	9374	9491	9608	9725	9842	9959	*0076	*0193	*0309	*0426
72	57 0543	0660	0776	0893	1010	1126	1243	1359	1476	1592
73	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755
74	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915
75	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072
76	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226
77	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377
78	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525
79	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669
380	57 9784	9898	*0012	*0126	*0241	*0355	*0469	*0583	*0697	*0811
81	58 0925	1039	1153	1267	1381	1495	1608	1722	1836	1950
82	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085
83	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218
84	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348
85	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475
86	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599
87	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720
88	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838
89	9950	*0061	*0173	*0284	*0396	*0507	*0619	*0730	*0842	*0953
390	59 1065	1176	1287	1399	1510	1621	1732	1843	1955	2066
91	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175
92	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282
93	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386
94	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487
95	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586
96	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681
97	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774
98	9883	9992	*0101	*0210	*0319	*0428	*0537	*0646	*0755	*0864
99	60 0973	1082	1191	1299	1408	1517	1625	1734	1843	1951
400	60 2060	2169	2277	2386	2494	2603	2711	2819	2928	3036

PROPORTIONAL PARTS

	124	123	122
1	12.4	12.3	12.2
2	24.8	24.6	24.4
3	37.2	36.9	36.6
4	49.6	49.2	48.8
5	62.0	61.5	61.0
6	74.4	73.8	73.2
7	86.8	86.1	85.4
8	99.2	98.4	97.6
9	111.6	110.7	109.8

	121	120	119
1	12.1	12.0	11.9
2	24.2	24.0	23.8
3	36.3	36.0	35.7
4	48.4	48.0	47.6
5	60.5	60.0	59.5
6	72.6	72.0	71.4
7	84.7	84.0	83.3
8	96.8	96.0	95.2
9	108.9	108.0	107.1

	118	117	116
1	11.8	11.7	11.6
2	23.6	23.4	23.2
3	35.4	35.1	34.8
4	47.2	46.8	46.4
5	59.0	58.5	58.0
6	70.8	70.2	69.6
7	82.6	81.9	81.2
8	94.4	93.6	92.8
9	106.2	105.3	104.4

	115	114	113
1	11.5	11.4	11.3
2	23.0	22.8	22.6
3	34.5	34.2	33.9
4	46.0	45.6	45.2
5	57.5	57.0	56.5
6	69.0	68.4	67.8
7	80.5	79.8	79.1
8	92.0	91.2	90.4
9	103.5	102.6	101.7

	112	111	110
1	11.2	11.1	11.0
2	22.4	22.2	22.0
3	33.6	33.3	33.0
4	44.8	44.4	44.0
5	56.0	55.5	55.0
6	67.2	66.6	66.0
7	78.4	77.7	77.0
8	89.6	88.8	88.0
9	100.8	99.9	99.0

	109	108
1	10.9	10.8
2	21.8	21.6
3	32.7	32.4
4	43.6	43.2
5	54.5	54.0
6	65.4	64.8
7	76.3	75.6
8	87.2	86.4
9	98.1	97.2

400–450 **TABLE I**

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
400	60 2060	2169	2277	2386	2494	2603	2711	2819	2928	3036
01	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118
02	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197
03	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274
04	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348
05	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419
06	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488
07	9594	9701	9808	9914	*0021	*0128	*0234	*0341	*0447	*0554
08	61 0660	0767	0873	0979	1086	1192	1298	1405	1511	1617
09	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678
410	61 2784	2890	2996	3102	3207	3313	3419	3525	3630	3736
11	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792
12	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845
13	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895
14	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943
15	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989
16	9093	9198	9302	9406	9511	9615	9719	9824	9928	*0032
17	62 0136	0240	0344	0448	0552	0656	0760	0864	0968	1072
18	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110
19	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146
420	62 3249	3353	3456	3559	3663	3766	3869	3973	4076	4179
21	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210
22	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238
23	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263
24	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287
25	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308
26	9410	9512	9613	9715	9817	9919	*0021	*0123	*0224	*0326
27	63 0428	0530	0631	0733	0835	0936	1038	1139	1241	1342
28	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356
29	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367
430	63 3468	3569	3670	3771	3872	3973	4074	4175	4276	4376
31	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383
32	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388
33	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390
34	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389
35	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387
36	9486	9586	9686	9785	9885	9984	*0084	*0183	*0283	*0382
37	64 0481	0581	0680	0779	0879	0978	1077	1177	1276	1375
38	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366
39	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354
440	64 3453	3551	3650	3749	3847	3946	4044	4143	4242	4340
41	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324
42	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306
43	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285
44	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262
45	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237
46	9335	9432	9530	9627	9724	9821	9919	*0016	*0113	*0210
47	65 0308	0405	0502	0599	0696	0793	0890	0987	1084	1181
48	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150
49	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116
450	653213	3309	3405	3502	3598	3695	3791	3888	3984	4080

PROPORTIONAL PARTS

	109	108	107
1	10.9	10.8	10.7
2	21.8	21.6	21.4
3	32.7	32.4	32.1
4	43.6	43.2	42.8
5	54.5	54.0	53.5
6	65.4	64.8	64.2
7	76.3	75.6	74.9
8	87.2	86.4	85.6
9	98.1	97.2	96.3

	106	105	104
1	10.6	10.5	10.4
2	21.2	21.0	20.8
3	31.8	31.5	31.2
4	42.4	42.0	41.6
5	53.0	52.5	52.0
6	63.6	63.0	62.4
7	74.2	73.5	72.8
8	84.8	84.0	83.2
9	95.4	94.5	93.6

	103	102	101
1	10.3	10.2	10.1
2	20.6	20.4	20.2
3	30.9	30.6	30.3
4	41.2	40.8	40.4
5	51.5	51.0	50.5
6	61.8	61.2	60.6
7	72.1	71.4	70.7
8	82.4	81.6	80.8
9	92.7	91.8	90.9

	100	99
1	10.0	9.9
2	20.0	19.8
3	30.0	29.7
4	40.0	39.6
5	50.0	49.5
6	60.0	59.4
7	70.0	69.3
8	80.0	79.2
9	90.0	89.1

	98
1	9.8
2	19.6
3	29.4
4	39.2
5	49.0
6	58.8
7	68.6
8	78.4
9	88.2

	97
1	9.7
2	19.4
3	29.1
4	38.8
5	48.5
6	58.2
7	67.9
8	77.6
9	87.3

450–500 **TABLE I**

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
450	65 3213	3309	3405	3502	3598	3695	3791	3888	3984	4080
51	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042
52	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002
53	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960
54	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916
55	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870
56	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821
57	9916	*0011	*0106	*0201	*0296	*0391	*0486	*0581	*0676	*0771
58	66 0865	0960	1055	1150	1245	1339	1434	1529	1623	1718
59	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663
460	66 2758	2852	2947	3041	3135	3230	3324	3418	3512	3607
61	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548
62	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487
63	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424
64	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360
65	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293
66	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224
67	9317	9410	9503	9596	9689	9782	9875	9967	*0060	*0153
68	67 0246	0339	0431	0524	0617	0710	0802	0895	0988	1080
69	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005
470	67 2098	2190	2283	2375	2467	2560	2652	2744	2836	2929
71	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850
72	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769
73	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687
74	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602
75	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516
76	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427
77	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337
78	9428	9519	9610	9700	9791	9882	9973	*0063	*0154	*0245
79	68 0336	0426	0517	0607	0698	0789	0879	0970	1060	1151
480	68 1241	1332	1422	1513	1603	1693	1784	1874	1964	2055
81	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957
82	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857
83	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756
84	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652
85	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547
86	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440
87	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331
88	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220
89	9309	9398	9486	9575	9664	9753	9841	9930	*0019	*0107
490	69 0196	0285	0373	0462	0550	0639	0728	0816	0905	0993
91	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877
92	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759
93	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639
94	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517
95	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394
96	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269
97	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142
98	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014
99	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883
500	69 8970	9057	9144	9231	9317	9404	9491	9578	9664	9751

PROPORTIONAL PARTS

	96	95	94
1	9.6	9.5	9.4
2	19.2	19.0	18.8
3	28.8	28.5	28.2
4	38.4	38.0	37.6
5	48.0	47.5	47.0
6	57.6	57.0	56.4
7	67.2	66.5	65.8
8	76.8	76.0	75.2
9	86.4	85.5	84.6

	93	92
1	9.3	9.2
2	18.6	18.4
3	27.9	27.6
4	37.2	36.8
5	46.5	46.0
6	55.8	55.2
7	65.1	64.4
8	74.4	73.6
9	83.7	82.8

	91	90
1	9.1	9.0
2	18.2	18.0
3	27.3	27.0
4	36.4	36.0
5	45.5	45.0
6	54.6	54.0
7	63.7	63.0
8	72.8	72.0
9	81.9	81.0

	89
1	8.9
2	17.8
3	26.7
4	35.6
5	44.5
6	53.4
7	62.3
8	71.2
9	80.1

	88
1	8.8
2	17.6
3	26.4
4	35.2
5	44.0
6	52.8
7	61.6
8	70.4
9	79.2

	87
1	8.7
2	17.4
3	26.1
4	34.8
5	43.5
6	52.2
7	60.9
8	69.6
9	78.3

500–550 **TABLE I**

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
500	69 8970	9057	9144	9231	9317	9404	9491	9578	9664	9751
01	9838	9924	*0011	*0098	*0184	*0271	*0358	*0444	*0531	*0617
02	70 0704	0790	0877	0963	1050	1136	1222	1309	1395	1482
03	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344
04	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205
05	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065
06	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922
07	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778
08	5864	5949	6035	6120	6206	6291	6376	6402	6547	6632
09	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485
510	70 7570	7655	7740	7826	7911	7996	8081	8166	8251	8336
11	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185
12	9270	9355	9440	9524	9609	9694	9779	9863	9948	*0033
13	71 0117	0202	0287	0371	0456	0540	0625	0710	0794	0879
14	0963	1048	1132	1217	130	1385	1470	1554	1639	1723
15	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566
16	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407
17	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246
18	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084
19	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920
520	71 6003	6087	6170	6254	6337	6421	6504	6588	6671	6754
21	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587
22	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419
23	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248
24	9331	9414	9497	9580	9663	9745	9828	9911	9994	*0077
25	72 0159	0242	0325	0407	0490	0573	0655	0738	0821	0903
26	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728
27	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552
28	2634	2716	2798	2831	2963	3045	3127	3209	3291	3374
29	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194
530	72 4276	4358	4440	4522	4604	4685	4767	4849	4931	5013
31	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830
32	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646
33	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460
34	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273
35	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084
36	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893
37	9974	*0055	*0136	*0217	*0298	*0378	*0459	*0540	*0621	*0702
38	73 0782	0863	0944	1024	1105	1186	1266	1347	1428	1508
39	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313
540	73 2394	2474	2555	2635	2715	2796	2876	2956	3037	3117
41	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919
42	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720
43	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519
44	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317
45	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113
46	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908
47	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701
48	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493
49	9572	9651	9731	9810	9889	9968	*0047	*0126	*0205	*0284
550	74 0363	0442	0521	0600	0678	0757	0836	0915	0994	1073

PROPORTIONAL PARTS

	87	86	85
1	8.7	8.6	8.5
2	17.4	17.2	17.0
3	26.1	25.8	25.5
4	34.8	34.4	34.0
5	43.5	43.0	42.5
6	52.2	51.6	51.0
7	60.9	60.2	59.5
8	69.6	68.8	68.0
9	78.3	77.4	76.5

	84	83
1	8.4	8.3
2	16.8	16.6
3	25.2	24.9
4	33.6	33.2
5	42.0	41.5
6	50.4	49.8
7	58.8	58.1
8	67.2	66.4
9	75.6	74.7

	82	81	80	79
1	8.2	8.1	8.0	7.9
2	16.4	16.2	16.0	15.8
3	24.6	24.3	24.0	23.7
4	32.8	32.4	32.0	31.6
5	41.0	40.5	40.0	39.5
6	49.2	48.6	48.0	47.4
7	57.4	56.7	56.0	55.3
8	65.6	64.8	64.0	63.2
9	73.8	72.9	72.0	71.1

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
550	74 0363	0442	0521	0600	0678	0757	0836	0915	0994	1073
51	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860
52	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647
53	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431
54	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215
55	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997
56	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777
57	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556
58	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334
59	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110
560	74 8188	8266	8343	8421	8498	8576	8653	8731	8808	8885
61	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659
62	9736	9814	9891	9968	*0045	*0123	*0200	*0277	*0354	*0431
63	75 0508	0586	0663	0740	0817	0894	0971	1048	1125	1202
64	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972
65	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740
66	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506
67	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272
68	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036
69	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799
570	75 5875	5951	6027	6103	6180	6256	6332	6408	6484	6560
71	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320
72	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079
73	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836
74	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592
75	9668	9743	9819	9894	9970	*0045	*0121	*0196	*0272	*0347
76	76 0422	0498	0573	0649	0724	0799	0875	0950	1025	1101
77	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853
78	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604
79	2679	2754	2829	2904	2978	3053	3128	3203	3278	3353
580	76 3428	3503	3578	3653	3727	3802	3877	3952	4027	4101
81	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848
82	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594
83	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338
84	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082
85	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823
86	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564
87	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303
88	9377	9451	9525	9599	9673	9746	9820	9894	9968	*0042
89	77 0115	0189	0263	0336	0410	0484	0557	0631	0705	0778
590	77 0852	0926	0999	1073	1146	1220	1293	1367	1440	1514
91	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248
92	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981
93	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713
94	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444
95	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173
96	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902
97	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629
98	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354
99	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079
600	77 8151	8224	8296	8368	8441	8513	8585	8658	8730	8802

PROPORTIONAL PARTS

	79	78
1	7.9	7.8
2	15.8	15.6
3	23.7	23.4
4	31.6	31.2
5	39.5	39.0
6	47.4	46.8
7	55.3	54.6
8	63.2	62.4
9	71.1	70.2

	77	76
1	7.7	7.6
2	15.4	15.2
3	23.1	22.8
4	30.8	30.4
5	38.5	38.0
6	46.2	45.6
7	53.9	53.2
8	61.6	60.8
9	69.3	68.4

	75	74	73	72
1	7.5	7.4	7.3	7.2
2	15.0	14.8	14.6	14.4
3	22.5	22.2	21.9	21.6
4	30.0	29.6	29.2	28.8
5	37.5	37.0	36.5	36.0
6	45.0	44.4	43.8	43.2
7	52.5	51.8	51.1	50.4
8	60.0	59.2	58.4	57.6
9	67.5	66.6	65.7	64.8

600–650 **TABLE I**

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
600	77 8151	8224	8296	8368	8441	8513	8585	8658	8730	8802
01	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524
02	9596	9669	9741	9813	9885	9957	*0029	*0101	*0173	*0245
03	78 0317	0389	0461	0533	0605	0677	0749	0821	0893	0965
04	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684
05	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401
06	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117
07	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832
08	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546
09	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259
610	78 5330	5401	5472	5543	5615	5686	5757	5828	5899	5970
11	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680
12	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390
13	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098
14	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804
15	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510
16	9581	9651	9722	9792	9863	9933	*0004	*0074	*0144	*0215
17	79 0285	0356	0426	0496	0567	0637	0707	0778	0848	0918
18	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620
19	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322
620	79 2392	2462	2532	2602	2672	2742	2812	2882	2952	3022
21	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721
22	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418
23	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115
24	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811
25	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505
26	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198
27	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890
28	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582
29	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272
630	79 9341	9409	9478	9547	9616	9685	9754	9823	9892	9961
31	80 0029	0098	0167	0236	0305	0373	0442	0511	0580	0648
32	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335
33	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021
34	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705
35	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389
36	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071
37	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753
38	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433
39	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112
640	80 6180	6248	6316	6384	6451	6519	6587	6655	6723	6790
41	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467
42	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143
43	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818
44	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492
45	9560	9627	9694	9762	9829	9896	9964	*0031	*0098	*0165
46	81 0233	0300	0367	0434	0501	0569	0636	0703	0770	0837
47	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508
48	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178
49	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847
650	81 2913	2980	3047	3114	3181	3247	3314	3381	3448	3514

PROPORTIONAL PARTS

	73	72
1	7.3	7.2
2	14.6	14.4
3	21.9	21.6
4	29.2	28.8
5	36.5	36.0
6	43.8	43.2
7	51.1	50.4
8	58.4	57.6
9	65.7	64.8

	71	70
1	7.1	7.0
2	14.2	14.0
3	21.3	21.0
4	28.4	28.0
5	35.5	35.0
6	42.6	42.0
7	49.7	49.0
8	56.8	56.0
9	63.9	63.0

	69
1	6.9
2	13.8
3	20.7
4	27.6
5	34.5
6	41.4
7	48.3
8	55.2
9	62.1

	68
1	6.8
2	13.6
3	20.4
4	27.2
5	34.0
6	40.8
7	47.6
8	54.4
9	61.2

	67
1	6.7
2	13.4
3	20.1
4	26.8
5	33.5
6	40.2
7	46.9
8	53.6
9	60.3

	66
1	6.6
2	13.2
3	19.8
4	26.4
5	33.0
6	39.6
7	46.2
8	52.8
9	59.4

650–700 TABLE I

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
650	81 2913	2980	3047	3114	3181	3247	3314	3381	3448	3514
51	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181
52	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847
53	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511
54	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175
55	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838
56	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499
57	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160
58	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820
59	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478
660	81 9544	9610	9676	9741	9807	9873	9939	*0004	*0070	*0136
61	82 0201	0267	0333	0399	0464	0530	0595	0661	0727	0792
62	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448
63	1514	1579	1645	1710	1775	1841	1906	1972	1037	2103
64	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756
65	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409
66	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061
67	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711
68	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361
69	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010
670	82 6075	6140	6204	6269	6334	6399	6464	6528	6593	6658
71	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305
72	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951
73	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595
74	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239
75	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882
76	9947	*0011	*0075	*0139	*0204	*0268	*0332	*0396	*0460	*0525
77	83 0589	0653	0717	0781	0845	0909	0973	1037	1102	1166
78	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806
79	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445
680	83 2509	2573	2637	2700	2764	2828	2892	2956	3020	3083
81	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721
82	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357
83	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993
84	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627
85	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261
86	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894
87	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525
88	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156
89	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786
690	83 8849	8912	8975	9038	9101	9164	9227	9289	9352	9415
91	9478	9541	9604	9667	9729	9792	9855	9918	9981	*0043
92	84 0106	0169	0232	0294	0357	0420	0482	0545	0608	0671
93	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297
94	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922
95	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547
96	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170
97	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793
98	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415
99	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036
700	84 5098	5160	5222	5284	5346	5408	5470	5532	5594	5656

Proportional Parts

	67	66	65	64	63	62
1	6.7	6.6	6.5	6.4	6.3	6.2
2	13.4	13.2	13.0	12.8	12.6	12.4
3	20.1	19.8	19.5	19.2	18.9	18.6
4	26.8	26.4	26.0	25.6	25.2	24.8
5	33.5	33.0	32.5	32.0	31.5	31.0
6	40.2	39.6	39.0	38.4	37.8	37.2
7	46.9	46.2	45.5	44.8	44.1	43.4
8	53.6	52.8	52.0	51.2	50.4	49.6
9	60.3	59.4	58.5	57.6	56.7	55.8

700–750 **TABLE I**

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
700	84 5098	5160	5222	5284	5346	5408	5470	5532	5594	5656
01	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275
02	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894
03	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511
04	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128
05	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743
06	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358
07	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972
08	85 0033	0095	0156	0217	0279	0340	0401	0462	0524	0585
09	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197
710	85 1258	1320	1381	1442	1503	1564	1625	1686	1747	1809
11	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419
12	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029
13	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637
14	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245
15	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852
16	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459
17	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064
18	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668
19	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272
720	85 7332	7393	7453	7513	7574	7634	7694	7755	7815	7875
21	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477
22	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078
23	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679
24	9739	9799	9859	9918	9978	*0038	*0098	*0158	*0218	*0278
25	86 0338	0398	0458	0518	0578	0637	0697	0757	0817	0877
26	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475
27	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072
28	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668
29	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263
730	86 3323	3382	3442	3501	3561	3620	3680	3739	3799	3858
31	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452
32	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045
33	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637
34	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228
35	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819
36	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409
37	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998
38	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586
39	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173
740	86 9232	9290	9349	9408	9466	9525	9584	9642	9701	9760
41	9818	9877	9935	9994	*0053	*0111	*0170	*0228	*0287	*0345
42	87 0404	0462	0521	0579	0638	0696	0755	0813	0872	0930
43	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515
44	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098
45	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681
46	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262
47	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844
48	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424
49	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003
750	87 5061	5119	5177	5235	5293	5351	5409	5466	5524	5582

PROPORTIONAL PARTS

	62	**61**	**60**	**59**	**58**	**57**
1	6.2	6.1	6.0	5.9	5.8	5.7
2	12.4	12.2	12.0	11.8	11.6	11.4
3	16.8	18.3	18.0	17.7	17.4	17.1
4	24.8	24.4	24.0	23.6	23.2	22.8
5	31.0	30.5	30.0	29.5	29.0	28.5
6	37.2	36.6	36.0	35.4	34.8	34.2
7	43.4	42.7	42.0	41.3	40.6	39.9
8	49.6	48.8	48.0	47.2	46.4	45.6
9	55.8	54.9	54.0	53.1	52.2	51.3

750–800 **TABLE I**

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
750	87 5061	5119	5177	5235	5293	5351	5409	5466	5524	5582
51	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160
52	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737
53	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314
54	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889
55	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464
56	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039
57	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612
58	9669	9726	9784	9841	9898	9956	*0013	*0070	*0127	*0185
59	88 0242	0299	0356	0413	0471	0528	0585	0642	0699	0756
760	88 0814	0871	0928	0985	1042	1099	1156	1213	1271	1328
61	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898
62	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468
63	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037
64	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605
65	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172
66	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739
67	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305
68	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870
69	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434
770	88 6491	6547	6604	6660	6716	6773	6829	6885	6942	6998
71	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561
72	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123
73	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685
74	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246
75	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806
76	9862	9918	9974	*0030	*0086	*0141	*0197	*0253	*0309	*0365
77	89 0421	0477	0533	0589	0645	0700	0756	0812	0868	0924
78	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482
79	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039
780	89 2095	2150	2206	2262	2317	2373	2429	2484	2540	2595
81	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151
82	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706
83	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261
84	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814
85	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367
86	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920
87	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471
88	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022
89	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572
790	89 7627	7682	7737	7792	7847	7902	7957	8012	8067	8122
91	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670
92	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218
93	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766
94	9821	9875	9930	9985	*0039	*0094	*0149	*0203	*0258	*0312
95	90 0367	0422	0476	0531	0586	0640	0695	0749	0804	0859
96	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404
97	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948
98	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492
99	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036
800	90 3090	3144	3199	3253	3307	3361	3416	3470	3524	3578

PROPORTIONAL PARTS

	58	**57**	**56**	**55**
1	5.8	5.7	5.6	5.5
2	11.6	11.4	11.2	11.0
3	17.4	17.1	16.8	16.5
4	23.2	22.8	22.4	22.0
5	29.0	28.5	28.0	27.5
6	34.8	34.2	33.6	33.0
7	40.6	39.9	39.2	38.5
8	46.4	45.6	44.8	44.0
9	52.2	51.3	50.4	49.5

800–850 **TABLE I**

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
800	90 3090	3144	3199	3253	3307	3361	3416	3470	3524	3578
01	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120
02	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661
03	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202
04	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742
05	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281
06	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820
07	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358
08	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895
09	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431
810	90 8485	8539	8592	8646	8699	8753	8807	8860	8914	8967
11	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503
12	9556	9610	9663	9716	9770	9823	9877	9930	9984	*0037
13	91 0091	0144	0197	0251	0304	0358	0411	0464	0518	0571
14	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104
15	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637
16	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169
17	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700
18	2753	2806	2839	2913	2966	3019	3072	3125	3178	3231
19	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761
820	91 3814	3867	3920	3973	4026	4079	4132	4184	4237	4290
21	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819
22	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347
23	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875
24	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401
25	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927
26	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453
27	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978
28	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502
29	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026
830	91 9078	9130	9183	9235	9287	9340	9392	9444	9496	9549
31	9601	9653	9706	9758	9810	9862	9914	9967	*0019	*0071
32	92 0123	0176	0228	0280	0332	0384	0436	0489	0541	0593
33	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114
34	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634
35	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154
36	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674
37	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192
38	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710
39	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228
840	92 4279	4331	4383	4434	4486	4538	4589	4641	4693	4744
41	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261
42	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776
43	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291
44	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805
45	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319
46	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832
47	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345
48	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857
49	8909	8959	9010	9061	9112	9163	9215	9266	9317	9368
850	92 9419	9470	9521	9572	9623	9674	9725	9776	9827	9879

PROPORTIONAL PARTS

	54	**53**	**52**	**51**
1	5.4	5.3	5.2	5.1
2	10.8	10.6	10.4	10.2
3	16.2	15.9	15.6	15.3
4	21.6	21.2	20.8	20.4
5	27.0	26.5	26.0	25.5
6	32.4	31.8	31.2	30.6
7	37.8	37.1	36.4	35.7
8	43.2	42.4	41.6	40.8
9	48.6	47.7	46.8	45.9

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
850	92 9419	9470	9521	9572	9623	9674	9725	9776	9827	9879
51	9930	9981	*0032	*0083	*0134	*0185	*0236	*0287	*0338	*0389
52	93 0440	0491	0542	0592	0643	0694	0745	0796	0847	0898
53	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407
54	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915
55	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423
56	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930
57	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437
58	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943
59	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448
860	93 4498	4549	4599	4650	4700	4751	4801	4852	4902	4953
61	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457
62	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960
63	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463
64	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966
65	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468
66	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969
67	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470
68	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970
69	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469
870	93 9519	9569	9619	9669	9719	9769	9819	9869	9918	9968
71	94 0018	0068	0118	0168	0218	0267	0317	0367	0417	0467
72	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964
73	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462
74	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958
75	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455
76	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950
77	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445
78	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939
79	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433
880	94 4483	4532	4581	4631	4680	4729	4779	4828	4877	4927
81	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419
82	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912
83	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403
84	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894
85	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385
86	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875
87	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364
88	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853
89	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341
890	94 9390	9439	9488	9536	9585	9634	9683	9731	9780	9829
91	9878	9926	9975	*0024	*0073	*0121	*0170	*0219	*0267	*0316
92	95 0365	0414	0462	0511	0560	0608	0657	0706	0754	0803
93	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289
94	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775
95	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260
96	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744
97	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228
98	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711
99	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194
900	95 4243	4291	4339	4387	4435	4484	4532	4580	4628	4677

PROPORTIONAL PARTS

	51	50	49	48
1	5.1	5.0	4.9	4.8
2	10.2	10.0	9.8	9.6
3	15.3	15.0	14.7	14.4
4	20.4	20.0	19.6	19.2
5	25.5	25.0	24.5	24.0
6	30.6	30.0	29.4	28.8
7	35.7	35.9	34.3	33.6
8	40.8	40.0	39.2	38.4
9	45.9	45.0	44.1	43.2

900–950 **TABLE I**

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
900	95 4243	4291	4339	4387	4435	4484	4532	4580	4628	4677
01	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158
02	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640
03	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120
04	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601
05	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080
06	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559
07	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038
08	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516
09	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994
910	95 9041	9089	9137	9185	9232	9280	9328	9375	9423	9471
11	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947
12	9995	*0042	*0090	*0138	*0185	*0233	*0280	*0328	*0376	*0423
13	96 0471	0518	0566	0613	0661	0709	0756	0804	0851	0899
14	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374
15	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848
16	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322
17	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795
18	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268
19	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741
920	96 3788	3835	3882	3929	3977	4024	4071	4118	4165	4212
21	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684
22	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155
23	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625
24	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095
25	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564
26	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033
27	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501
28	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969
29	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436
930	96 8483	8530	8576	8623	8670	8716	8763	8810	8856	8903
31	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369
32	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835
33	9882	9928	9975	*0021	*0068	*0114	*0161	*0207	*0254	*0300
34	97 0347	0393	0440	0486	0533	0579	0626	0672	0719	0765
35	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229
36	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693
37	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157
38	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619
39	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082
940	97 3128	3174	3220	3266	3313	3359	3405	3451	3497	3543
41	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005
42	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466
43	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926
44	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386
45	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845
46	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304
47	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763
48	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220
49	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678
950	97 7724	7769	7815	7861	7906	7952	7998	8043	8089	8135

PROPORTIONAL PARTS

	48	47	46
1	4.8	4.7	4.6
2	9.6	9.4	9.2
3	14.4	14.1	13.8
4	19.2	18.8	18.4
5	24.0	23.5	23.0
6	28.8	28.2	27.6
7	33.6	32.9	32.2
8	38.4	37.6	36.8
9	43.2	42.3	41.4

950–1000 **TABLE I**

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
950	97 7724	7769	7815	7861	7906	7952	7998	8043	8089	8135
51	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591
52	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047
53	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503
54	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958
55	98 0003	0049	0094	0140	0185	0231	0276	0322	0367	0412
56	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867
57	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320
58	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773
59	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226
960	98 2271	2316	2362	2407	2452	2497	2543	2588	2633	2678
61	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130
62	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581
63	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032
64	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482
65	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932
66	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382
67	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830
68	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279
69	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727
970	98 6772	6817	6861	6906	6951	6996	7040	7085	7130	7175
71	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622
72	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068
73	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514
74	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960
75	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405
76	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850
77	9895	9939	9983	*0028	*0072	*0117	*0161	*0206	*0250	*0294
78	99 0339	0383	0428	0472	0516	0561	0605	0650	0694	0738
79	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182
980	99 1226	1270	1315	1359	1403	1448	1492	1336	1580	1625
81	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067
82	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509
83	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951
84	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392
85	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833
86	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273
87	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713
88	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152
89	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591
990	99 5635	5679	5723	5767	5811	5854	5898	5942	5986	6030
91	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468
92	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906
93	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343
94	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779
95	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216
96	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652
97	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087
98	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522
99	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957
1000										

PROPORTIONAL PARTS

	45
1	4.5
2	9.0
3	13.5
4	18.0
5	22.5
6	27.0
7	31.5
8	36.0
9	40.5

	44
1	4.4
2	8.8
3	13.2
4	17.6
5	22.0
6	26.4
7	30.8
8	35.2
9	39.9

	43
1	4.3
2	8.6
3	12.9
4	17.2
5	21.5
6	25.8
7	30.1
8	34.4
9	38.7

TABLE I (Supplement)

1000–1050 LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
1000	000 0000	0434	0869	1303	1737	2171	2605	3039	3473	3907
1001	4341	4775	5208	5642	6076	6510	6943	7377	7810	8244
1002	8677	9111	9544	9977	*0411	*0844	*1277	*1710	*2143	*2576
1003	001 3009	3442	3875	4308	4741	5174	5607	6039	6472	6905
1004	7337	7770	8202	8635	9067	9499	9932	*0364	*0796	*1228
1005	002 1661	2093	2525	2957	3389	3821	4253	4685	5116	5548
1006	5980	6411	6843	7275	7706	8138	8569	9001	9432	9863
1007	003 0295	0726	1157	1588	2019	2451	2882	3313	3744	4174
1008	4605	5036	5467	5898	6328	6759	7190	7620	8051	8481
1009	8912	9342	9772	*0203	*0633	*1063	*1493	*1924	*2354	*2784
1010	004 3214	3644	4074	4504	4933	5363	5793	6223	6652	7082
1011	7512	7941	8371	8800	9229	9659	*0088	*0517	*0947	*1376
1012	005 1805	2234	2663	3092	3521	3950	4379	4808	5237	5666
1013	6094	6523	6952	7380	7809	8238	8666	9094	9523	9951
1014	006 0380	0808	1236	1664	2092	2521	2949	3377	3805	4233
1015	4660	5088	5516	5944	6372	6799	7227	7655	8082	8510
1016	8937	9365	9792	*0219	*0647	*1074	*1501	*1928	*2355	*2782
1017	007 3210	3637	4064	4490	4917	5344	5771	6198	6624	7051
1018	7478	7904	8331	8757	9184	9610	*0037	*0463	*0889	*1316
1019	008 1742	2168	2594	3020	3446	3872	4298	4724	5150	5576
1020	6002	6427	6853	7279	7704	8130	8556	8981	9407	9832
1021	009 0257	0683	1108	1533	1959	2384	2809	3234	3659	4084
1022	4509	4934	5359	5784	6208	6633	7058	7483	7907	8332
1023	8756	9181	9605	*0030	*0454	*0878	*1303	*1727	* 2151	*2575
1024	010 3000	3424	3848	4272	4696	5120	5544	5967	6391	6815
1025	7239	7662	8086	8510	8933	9357	9780	*0204	*0627	*1050
1026	011 1474	1897	2320	2743	3166	3590	4013	4436	4859	5282
1027	5704	6127	6550	6973	7396	7818	8241	8664	9086	9509
1028	9931	*0354	*0776	*1198	*1621	*2043	*2465	*2887	*3310	*3732
1029	012 4154	4576	4998	5420	5842	6264	6685	7107	7529	7951
1030	8372	8794	9215	9637	*0059	*0480	*0901	*1323	*1744	*2165
1031	013 2587	3008	3429	3850	4271	4692	5113	5534	5955	6376
1032	6797	7218	7639	8059	8480	8901	9321	9742	*0162	*0583
1033	014 1003	1424	1844	2264	2685	3105	3525	3945	4365	4785
1034	5205	5625	6045	6465	6885	7305	7725	8144	8564	8984
1035	9403	9823	*0243	*0662	*1082	*1501	*1920	*2340	*2759	*3178
1036	015 3598	4017	4436	4855	5274	5693	6112	6531	6950	7369
1037	7788	8206	8625	9044	9462	9881	*0300	*0718	*1137	*1555
1038	016 1974	2392	2810	3229	3647	4065	4483	4901	5319	5737
1039	6155	6573	6991	7409	7827	8245	8663	9080	9498	9916
1040	017 0333	0751	1168	1586	2003	2421	2838	3256	3673	4090
1041	4507	4924	5342	5759	6176	6593	7010	7427	7844	8260
1042	8677	9094	9511	9927	*0344	*0761	*1177	*1594	*2010	*2427
1043	018 2843	3259	3676	4092	4508	4925	5341	5757	6173	6589
1044	7005	7421	7837	8253	8669	9084	9500	9916	*0332	*0747
1045	019 1163	1578	1994	2410	2825	3240	3656	4071	4486	4902
1046	5317	5732	6147	6562	6977	7392	7807	8222	8637	9052
1047	9467	9882	*0296	*0711	*1126	*1540	*1955	*2369	*2784	*3198
1048	020 3613	4027	4442	4856	5270	5684	6099	6513	6927	7341
1049	7755	8169	8583	8997	9411	9824	*0238	*0652	*1066	*1479
1050	021 1893	2307	2720	3134	3547	3961	4374	4787	5201	5614

TABLE I (Supplement)

1050–1100 **LOGARITHMS**

N	0	1	2	3	4	5	6	7	8	9
1050	021 1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
1051	6027	6440	6854	7267	7680	8093	8506	8919	9332	9745
1052	022 0157	0570	0983	1396	1808	2221	2634	3046	3459	3871
1053	4284	4696	5109	5521	5933	6345	6758	7170	7582	7994
1054	8406	8818	9230	9642	*0054	*0466	*0878	*1289	*1701	*2113
1055	023 2525	2936	3348	3759	4171	4582	4994	5405	5817	6228
1056	6639	7050	7462	7873	8284	8695	9106	9517	9928	*0339
1057	024 0750	1161	1572	1982	2393	2804	3214	3625	4036	4446
1058	4857	5267	5678	6088	6498	6909	7319	7729	8139	8549
1059	8960	9370	9780	*0190	*0600	*1010	*1419	*1829	*2239	*2649
1060	025 3059	3468	3878	4288	4697	5107	5516	5926	6335	6744
1061	7154	7563	7972	8382	8791	9200	9609	*0018	*0427	*0836
1062	026 1245	1654	2063	2472	2881	3289	3698	4107	4515	4924
1063	5333	5741	6150	6558	6967	7375	7783	8192	8600	9008
1064	9416	9824	*0233	*0641	*1049	*1457	*1865	*2273	*2680	*3088
1065	027 3496	3904	4312	4719	5127	5535	5942	6350	6757	7165
1066	7572	7979	8387	8794	9201	9609	*0016	*0423	*0830	*1237
1067	028 1644	2051	2458	2865	3272	3679	4086	4492	4899	5306
1068	5713	6119	6526	6932	7339	7745	8152	8558	8964	9371
1069	9777	*0183	*0590	*0996	*1402	*1808	*2214	*2620	*3026	*3432
1070	029 3838	4244	4649	5055	5461	5867	6272	6678	7084	7489
1071	7895	8300	8706	9111	9516	9922	*0327	*0732	*1138	*1543
1072	030 1948	2353	2758	3163	3568	3973	4378	4783	5188	5592
1073	5997	6402	6807	7211	7616	8020	8425	8830	9234	9638
1074	031 0043	0447	0851	1256	1660	2064	2468	2872	3277	3681
1075	4085	4489	4893	5296	5700	6104	6508	6912	7315	7719
1076	8123	8526	8930	9333	9737	*0140	*0544	*0947	*1350	*1754
1077	032 2157	2560	2963	3367	3770	4173	4576	4979	5382	5785
1078	6188	6590	6993	7396	7799	8201	8604	9007	9409	9812
1079	033 0214	0617	1019	1422	1824	2226	2629	3031	3433	3835
1080	4238	4640	5042	5444	5846	6248	6650	7052	7453	7855
1081	8257	8659	9060	9462	9864	*0265	*0667	*1068	*1470	*1871
1082	034 2273	2674	3075	3477	3878	4279	4680	5081	5482	5884
1083	6285	6686	7087	7487	7888	8289	8690	9091	9491	9892
1084	035 0293	0693	1094	1495	1895	2296	2696	3096	3497	3897
1085	4297	4698	5098	5498	5898	6298	6698	7098	7498	7898
1086	8298	8698	9098	9498	9898	*0297	*0697	*1097	*1496	*1896
1087	036 2295	2695	3094	3494	3893	4293	4692	5091	5491	5890
1088	6289	6688	7087	7486	7885	8284	8683	9082	9481	9880
1089	037 0279	0678	1076	1475	1874	2272	2671	3070	3468	3867
1090	4265	4663	5062	5460	5858	6257	6655	7053	7451	7849
1091	8248	8646	9044	9442	9839	*0237	*0635	*1033	*1431	*1829
1092	038 2226	2624	3022	3419	3817	4214	4612	5009	5407	5804
1093	6202	6599	6996	7393	7791	8188	8585	8982	9379	9776
1094	039 0173	0570	0967	1364	1761	2158	2554	2951	3348	3745
1095	4141	4538	4934	5331	5727	6124	6520	6917	7313	7709
1096	8106	8502	8898	9294	9690	*0086	*0482	*0878	*1274	*1670
1097	040 2066	2462	2858	3254	3650	4045	4441	4837	5232	5628
1098	6023	6419	6814	7210	7605	8001	8396	8791	9187	9582
1099	9977	*0372	*0767	*1162	*1557	*1952	*2347	*2342	*3137	*3532
1100	041 3927	4322	4716	5111	5506	5900	6295	6690	7084	7479

TABLE II. The Number of Each Day of the Year Counting from January 1

Day of Month	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Day of Month
1	1	32	60	91	121	152	182	213	244	274	305	335	**1**
2	2	33	61	92	122	153	183	214	245	275	306	336	**2**
3	3	34	62	93	123	154	184	215	246	276	307	337	**3**
4	4	35	63	94	124	155	185	216	247	277	308	338	**4**
5	5	36	64	95	125	156	186	217	248	278	309	339	**5**
6	6	37	65	96	126	157	187	218	249	279	310	340	**6**
7	7	38	66	97	127	158	188	219	250	280	311	341	**7**
8	8	39	67	98	128	159	189	220	251	281	312	342	**8**
9	9	40	68	99	129	160	190	221	252	282	313	343	**9**
10	10	41	69	100	130	161	191	222	253	283	314	344	**10**
11	11	42	70	101	131	162	192	223	254	284	315	345	**11**
12	12	43	71	102	132	163	193	224	255	285	316	346	**12**
13	13	44	72	103	133	164	194	225	256	286	317	347	**13**
14	14	45	73	104	134	165	195	226	257	287	318	348	**14**
15	15	46	74	105	135	166	196	227	258	288	319	349	**15**
16	16	47	75	106	136	167	197	228	259	289	320	350	**16**
17	17	48	76	107	137	168	198	229	260	290	321	351	**17**
18	18	49	77	108	138	169	199	230	261	291	322	352	**18**
19	19	50	78	109	139	170	200	231	262	292	323	353	**19**
20	20	51	79	110	140	171	201	232	263	293	324	354	**20**
21	21	52	80	111	141	172	202	233	264	294	325	355	**21**
22	22	53	81	112	142	173	203	234	265	295	326	356	**22**
23	23	54	82	113	143	174	204	235	266	296	327	357	**23**
24	24	55	83	114	144	175	205	236	267	297	328	358	**24**
25	25	56	84	115	145	176	206	237	268	298	329	359	**25**
26	26	57	85	116	146	177	207	238	269	299	330	360	**26**
27	27	58	86	117	147	178	208	239	270	300	331	361	**27**
28	28	59	87	118	148	179	209	240	271	301	332	362	**28**
29	29		88	119	149	180	210	241	272	302	333	363	**29**
30	30		89	120	150	181	211	242	273	303	334	364	**30**
31	31		90		151		212	243		304		365	**31**

Note. — In leap years, after February 28, add 1 to the tabular number.

TABLE III. The Amount of 1 at Compound Interest

$$(1+i)^n$$

n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{8}\%$	1%
1	1.0033 3333	1.0041 6667	1.0050 0000	1.0087 5000	1.0100 0000
2	1.0066 7778	1.0083 5069	1.0100 2500	1.0175 7656	1.0201 0000
3	1.0100 3337	1.0125 5216	1.0150 7513	1.0264 8036	1.0303 0100
4	1.0134 0015	1.0167 7112	1.0201 5050	1.0354 6206	1.0406 0401
5	1.0167 7815	1.0210 0767	1.0252 5125	1.0445 2235	1.0510 1005
6	1.0201 6741	1.0252 6187	1.0303 7751	1.0536 6192	1.0615 2015
7	1.0235 6797	1.0295 3379	1.0355 2940	1.0628 8147	1.0721 3535
8	1.0269 7986	1.0338 2352	1.0407 0704	1.0721 8168	1.0828 5671
9	1.0304 0313	1.0381 3111	1.0459 1058	1.0815 6327	1.0936 8527
10	1.0338 3780	1.0424 5666	1.0511 4013	1.0910 2695	1.1046 2213
11	1.0372 8393	1.0468 0023	1.0563 9583	1.1005 7343	1.1156 6835
12	1.0407 4154	1.0511 6190	1.0616 7781	1.1102 0345	1.1268 2503
13	1.0442 1068	1.0555 4174	1.0669 8620	1.1199 1773	1.1380 9328
14	1.0476 9138	1.0599 3983	1.0723 2113	1.1297 1701	1.1494 7421
15	1.0511 8369	1.0643 5625	1.0776 8274	1.1396 0203	1.1609 6896
16	1.0546 8763	1.0687 9106	1.0830 7115	1.1495 7355	1.1725 7864
17	1.0582 0326	1.0732 4436	1.0884 8651	1.1596 3232	1.1843 0443
18	1.0617 3060	1.0777 1621	1.0939 2894	1.1697 7910	1.1961 4748
19	1.0652 6971	1.0822 0670	1.0993 9858	1.1800 1467	1.2081 0895
20	1.0688 2060	1.0867 1589	1.1048 9558	1.1903 3980	1.2201 9004
21	1.0723 8334	1.0912 4387	1.1104 2006	1.2007 5527	1.2323 9194
22	1.0759 5795	1.0957 9072	1.1159 7216	1.2112 6188	1.2447 1586
23	1.0795 4448	1.1003 5652	1.1215 5202	1.2218 6042	1.2571 6302
24	1.0831 4296	1.1049 4134	1.1271 5978	1.2325 5170	1.2697 3465
25	1.0867 5344	1.1095 4526	1.1327 9558	1.2433 3653	1.2824 3200
26	1.0903 7595	1.1141 6836	1.1384 5955	1.2542 1572	1.2952 5631
27	1.0940 1053	1.1188 1073	1.1441 5185	1.2651 9011	1.3082 0888
28	1.0976 5724	1.1234 7244	1.1498 7261	1.2762 6052	1.3212 9097
29	1.1013 1609	1.1281 5358	1.1556 2197	1.2874 2780	1.3345 0388
30	1.1049 8715	1.1328 5422	1.1614 0008	1.2986 9280	1.3478 4892
31	1.1086 7044	1.1375 7444	1.1672 0708	1.3100 5636	1.3613 2740
32	1.1123 6601	1.1423 1434	1.1730 4312	1.3215 1935	1.3749 4068
33	1.1160 7389	1.1470 7398	1.1789 0833	1.3330 8265	1.3886 9009
34	1.1197 9414	1.1518 5346	1.1848 0288	1.3447 4712	1.4025 7699
35	1.1235 2679	1.1566 5284	1.1907 2689	1.3565 1366	1.4166 0276
36	1.1272 7187	1.1614 7223	1.1966 8052	1.3683 8315	1.4307 6878
37	1.1310 2945	1.1663 1170	1.2026 6393	1.3803 5650	1.4450 7647
38	1.1347 9955	1.1711 7133	1.2086 7725	1.3924 3462	1.4595 2724
39	1.1385 8221	1.1760 5121	1.2147 2063	1.4046 1843	1.4741 2251
40	1.1423 7748	1.1809 5142	1.2207 9424	1.4146 0884	1.4888 6373
41	1.1461 8541	1.1858 7206	1.2268 9821	1.4293 0679	1.5037 5237
42	1.1500 0603	1.1908 1319	1.2330 3270	1.4418 1322	1.5187 8989
43	1.1538 3938	1.1957 7491	1.2391 9786	1.4544 2909	1.5339 7779
44	1.1576 8551	1.2007 5731	1.2453 9385	1.4671 5534	1.5493 1757
45	1.1615 4446	1.2057 6046	1.2516 2082	1.4799 9295	1.5648 1075
46	1.1654 1628	1.2107 8446	1.2578 7892	1.4929 4289	1.5804 5885
47	1.1693 0100	1.2158 2940	1.2641 6832	1.5060 0614	1.5962 6344
48	1.1731 9867	1.2208 9536	1.2704 8916	1.5191 8370	1.6122 2608
49	1.1771 0933	1.2259 8242	1.2768 4161	1.5324 7655	1.6283 4834
50	1.1810 3303	1.2310 9068	1.2832 2581	1.5458 8572	1.6446 3182

TABLE III. The Amount of 1 at Compound Interest

$(1+i)^n$

n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{8}\%$	1%
51	1.1849 6981	1.2362 2002	1.2896 4194	1.5594 1222	1.6610 7814
52	1.1889 1971	1.2413 7114	1.2960 9015	1.5730 5708	1.6776 8892
53	1.1928 8277	1.2465 4352	1.3025 7060	1.5868 2133	1.6944 6581
54	1.1968 5905	1.2517 3745	1.3090 8346	1.6007 0602	1.7114 1047
55	1.2008 4858	1.2569 5302	1.3156 2887	1.6147 1219	1.7285 2457
56	1.2048 5141	1.2621 9033	1.3222 0702	1.6288 4093	1.7458 0982
57	1.2088 6758	1.2674 4946	1.3288 1805	1.6430 9328	1.7632 6792
58	1.2128 9714	1.2727 3050	1.3354 6214	1.6574 7035	1.7809 0060
59	1.2169 4013	1.2780 3354	1.3421 3946	1.6719 7322	1.7987 0960
60	1.2209 9659	1.2833 5868	1.3488 5015	1.6866 0298	1.8166 9670
61	1.2250 6658	1.2887 0601	1.3555 9440	1.7013 6076	1.8348 6367
62	1.2291 5014	1.2940 7561	1.3623 7238	1.7162 4766	1.8532 1230
63	1.2332 4730	1.2994 6760	1.3691 8424	1.7312 6483	1.8717 4443
64	1.2373 6813	1.3048 8204	1.3760 3016	1.7464 1340	1.8904 6187
65	1.2414 8266	1.3103 1905	1.3829 1031	1.7616 9452	1.9093 6649
66	1.2456 2093	1.3157 7872	1.3898 2486	1.7771 0934	1.9284 6015
67	1.2497 7300	1.3212 6113	1.3967 7399	1.7926 5905	1.9477 4475
68	1.2539 3891	1.3267 6638	1.4037 5785	1.8083 4482	1.9672 2220
69	1.2581 1871	1.3322 9458	1.4107 7664	1.8241 6783	1.9868 9442
70	1.2623 1244	1.3378 4580	1.4178 3053	1.8401 2930	2.0067 6337
71	1.2665 2015	1.3434 2016	1.4249 1968	1.8562 3043	2.0268 3100
72	1.2707 4188	1.3490 1774	1.4320 4428	1.8724 7245	2.0470 9931
73	1.2749 7769	1.3546 3865	1.4392 0450	1.8888 5658	2.0675 7031
74	1.2792 2761	1.3602 8298	1.4464 0052	1.9053 8408	2.0882 4601
75	1.2834 9170	1.3659 5082	1.4536 3252	1.9220 5619	2.1091 2847
76	1.2877 7001	1.3716 4229	1.4609 0069	1.9388 7418	2.1302 1975
77	1.2920 6258	1.3773 5746	1.4682 0519	1.9558 3933	2.1515 2195
78	1.2963 6945	1.3830 9645	1.4755 4622	1.9729 5292	2.1730 3717
79	1.3006 9068	1.3888 5935	1.4829 2395	1.9902 1626	2.1947 6754
80	1.3050 2632	1.3946 4627	1.4903 3857	2.0076 3066	2.2167 1522
81	1.3093 7641	1.4004 5729	1.4977 9026	2.0251 9742	2.2388 8237
82	1.3137 4099	1.4062 9253	1.5052 7921	2.0429 1790	2.2612 7119
83	1.3181 2013	1.4121 5209	1.5128 0561	2.0607 9343	2.2838 8390
84	1.3225 1386	1.4180 3605	1.5203 6964	2.0788 2537	2.3067 2274
85	1.3269 2224	1.4239 4454	1.5279 7148	2.0970 1510	2.3297 8997
86	1.3313 4532	1.4298 7764	1.5356 1134	2.1153 6398	2.3530 8787
87	1.3357 8314	1.4358 3546	1.5432 8940	2.1338 7341	2.3766 1875
88	1.3402 3575	1.4418 1811	1.5510 0585	2.1525 4481	2.4003 8494
89	1.3447 0320	1.4478 2568	1.5587 6087	2.1713 7957	2.4243 8879
90	1.3491 8554	1.4538 5829	1.5665 5468	2.1903 7914	2.4486 3267
91	1.3536 8283	1.4599 1603	1.5743 8745	2.2095 4496	2.4731 1900
92	1.3581 9510	1.4659 9902	1.5822 5939	2.2288 7848	2.4978 5019
93	1.3627 2242	1.4721 0735	1.5901 7069	2.2483 8117	2.5228 2869
94	1.3672 6483	1.4782 4113	1.5981 2154	2.2680 5450	2.5480 5698
95	1.3718 2238	1.4844 0047	1.6061 1215	2.2878 9998	2.5735 3755
96	1.3763 9512	1.4905 8547	1.6141 4271	2.3079 1910	2.5992 7293
97	1.3809 8310	1.4967 9624	1.6222 1342	2.3281 1340	2.6252 6565
98	1.3855 8638	1.5030 3289	1.6303 2449	2.3484 8439	2.6515 1831
99	1.3902 0500	1.5092 9553	1.6384 7611	2.3690 3363	2.6780 3349
100	1.3948 3902	1.5155 8426	1.6466 6849	2.3862 6267	2.7048 1383

TABLE III. The Amount of 1 at Compound Interest

$(1+i)^n$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
1	1.0112 5000	1.0125 0000	1.0137 5000	1.0150 0000	1.0175 0000
2	1.0226 2656	1.0251 5625	1.0276 8906	1.0302 2500	1.0353 0625
3	1.0341 3111	1.0379 7070	1.0418 1979	1.0456 7838	1.0534 2411
4	1.0457 6509	1.0509 4534	1.0561 4481	1.0613 6355	1.0718 5903
5	1.0575 2994	1.0640 8215	1.0706 6680	1.0772 8400	1.0906 1656
6	1.0694 2716	1.0773 8318	1.0853 8847	1.0934 4326	1.1097 0235
7	1.0814 5821	1.0908 5047	1.1003 1256	1.1098 4491	1.1291 2215
8	1.0936 2462	1.1044 8610	1.1154 4186	1.1264 9259	1.1488 8178
9	1.1059 2789	1.1182 9218	1.1307 7918	1.1433 8998	1.1689 8721
10	1.1183 6958	1.1322 7083	1.1463 2740	1.1605 4083	1.1894 4449
11	1.1309 5124	1.1464 2422	1.1620 8940	1.1779 4894	1.2102 5977
12	1.1436 7444	1.1607 5452	1.1780 6813	1.1956 1817	1.2314 3931
13	1.1565 4078	1.1752 6395	1.1942 6656	1.2135 5244	1.2529 8950
14	1.1695 5186	1.1899 5475	1.2106 8773	1.2317 5573	1.2749 1682
15	1.1827 0932	1.2048 2918	1.2273 3469	1.2502 3207	1.2972 2786
16	1.1960 1480	1.2198 8955	1.2442 1054	1.2689 8555	1.3199 2935
17	1.2094 6997	1.2351 3817	1.2613 1843	1.2880 2033	1.3430 2811
18	1.2230 7650	1.2505 7739	1.2786 6156	1.3073 4064	1.3665 3111
19	1.2368 3611	1.2662 0961	1.2962 4316	1.3269 5075	1.3904 4540
20	1.2507 5052	1.2820 3723	1.3140 6650	1.3468 5501	1.4147 7820
21	1.2648 2143	1.2980 6270	1.3321 3492	1.3670 5783	1.4395 3681
22	1.2790 5071	1.3142 8848	1.3504 5177	1.3875 6370	1.4647 2871
23	1.2934 4003	1.3307 1709	1.3690 2048	1.4083 7715	1.4903 6146
24	1.3079 9123	1.3473 5105	1.3878 4451	1.4295 0281	1.5164 4279
25	1.3227 0613	1.3641 9294	1.4069 2738	1.4509 4535	1.5429 8054
26	1.3375 8657	1.3812 4535	1.4262 7263	1.4727 0953	1.5699 8269
27	1.3526 3442	1.3985 1092	1.4458 8388	1.4948 0018	1.5974 5739
28	1.3678 5156	1.4159 9230	1.4657 6478	1.5172 2218	1.6254 1290
29	1.3832 3989	1.4336 9221	1.4859 1905	1.5399 8051	1.6538 5762
30	1.3988 0134	1.4516 1336	1.5063 5043	1.5630 8022	1.6828 0013
31	1.4145 3785	1.4697 5853	1.5270 6275	1.5865 2642	1.7122 4913
32	1.4304 5140	1.4881 3051	1.5480 5986	1.6103 2432	1.7422 1349
33	1.4465 4398	1.5067 3214	1.5693 5469	1.6344 7918	1.7727 0223
34	1.4628 1760	1.5255 6629	1.5909 2419	1.6589 9637	1.8037 2452
35	1.4792 7430	1.5446 3587	1.6127 9940	1.6838 8132	1.8352 8970
36	1.4959 1613	1.5639 4382	1.6349 7539	1.7091 3954	1.8674 0727
37	1.5127 4519	1.5834 9312	1.6574 5630	1.7347 7663	1.9000 8689
38	1.5297 6357	1.6032 8678	1.6802 4633	1.7607 9828	1.9333 3841
39	1.5469 7341	1.6233 2787	1.7033 4971	1.7872 1025	1.9671 7184
40	1.5643 7687	1.6436 1946	1.7267 7077	1.8140 1841	2.0015 9734
41	1.5819 7611	1.6641 6471	1.7505 1387	1.8412 2868	2.0366 2530
42	1.5997 7334	1.6849 6677	1.7745 8343	1.8688 4712	2.0722 6624
43	1.6177 7079	1.7060 2885	1.7989 8396	1.8968 7982	2.1085 3090
44	1.6359 7071	1.7273 5421	1.8237 1999	1.9253 3302	2.1454 3019
45	1.6543 7538	1.7489 4614	1.8487 9614	1.9542 1301	2.1829 7522
46	1.6729 8710	1.7708 0797	1.8742 1708	1.9835 2621	2.2211 7728
47	1.6918 0821	1.7929 4306	1.8999 8757	2.0132 7910	2.2600 4789
48	1.7108 4105	1.8153 5485	1.9261 1240	2.0434 7829	2.2995 9872
49	1.7300 8801	1.8380 4679	1.9525 9644	2.0741 3046	2.3398 4170
50	1.7495 5150	1.8610 2237	1.9794 4464	2.1052 4242	2.3807 8803

TABLE III. The Amount of 1 at Compound Interest

$(1+i)^n$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
51	1.7692 3395	1.8842 8515	2.0066 6201	2.1368 2106	2.4224 5274
52	1.7891 3784	1.9078 3872	2.0342 5361	2.1688 7337	2.4648 4566
53	1.8092 6564	1.9316 8670	2.0622 2460	2.2014 0647	2.5079 8046
54	1.8296 1988	1.9558 3279	2.0905 8019	2.2344 2757	2.5518 7012
55	1.8502 0310	1.9802 8070	2.1193 2566	2.2679 4398	2.5965 2785
56	1.8710 1788	2.0050 3420	2.1484 6639	2.3019 6314	2.6419 6708
57	1.8920 6684	2.0300 9713	2.1780 0780	2.3364 9259	2.6882 0151
58	1.9133 5259	2.0554 7335	2.2079 5541	2.3715 3998	2.7352 4503
59	1.9348 7780	2.0811 6676	2.2383 1480	2.4071 1308	2.7831 1182
60	1.9566 4518	2.1071 8135	2.2690 9163	2.4432 1978	2.8318 1628
61	1.9786 5744	2.1335 2111	2.3002 9164	2.4798 6807	2.8813 7306
62	2.0009 1733	2.1601 9013	2.3319 2065	2.5170 6609	2.9317 9709
63	2.0234 2765	2.1871 9250	2.3639 8456	2.5548 2208	2.9831 0354
64	2.0461 9121	2.2145 3241	2.3964 8934	2.5931 4442	3.0343 0785
65	2.6092 1087	2.2422 1407	2.4294 4107	2.6320 4158	3.0884 2574
66	2.0924 8949	2.2702 4174	2.4628 4589	2.6715 2221	3.1424 7319
67	2.1160 2999	2.2986 1976	2.4967 1002	2.7115 9504	3.1974 6647
68	2.1398 3533	2.3273 5251	2.5310 3978	2.7522 6896	3.2534 2213
69	2.1639 0848	2.3564 4442	2.5658 4158	2.7935 5300	3.3103 5702
70	2.1882 5245	2.3858 9997	2.6011 2190	2.8354 5629	3.3682 8827
71	2.2128 7029	2.4157 2372	2.6368 8732	2.8779 8814	3.4272 3331
72	2.2377 6508	2.4459 2027	2.6731 4453	2.9211 5796	3.4872 0990
73	2.2629 3994	2.4764 9427	2.7099 0026	2.9649 7533	3.5482 3607
74	2.2833 9801	2.5074 5045	2.7471 6139	3.0094 4996	3.6103 3020
75	2.3141 4249	2.5387 9358	2.7849 3486	3.0545 9171	3.6735 1098
76	2.3401 7659	2.5705 2850	2.8232 2771	3.1004 1059	3.7377 9742
77	2.3665 0358	2.6026 6011	2.8620 4710	3.1469 1674	3.8032 0888
78	2.3931 2675	2.6351 9336	2.9014 0024	3.1941 2050	3.8697 6503
79	2.4200 4942	2.6681 3327	2.9412 9450	3.2420 3230	3.9374 8592
80	2.4472 7498	2.7014 8494	2.9817 3730	3.2906 6279	4.0063 9192
81	2.4748 0682	2.7352 5350	3.0227 3618	3.3400 2273	4.0765 0378
82	2.5026 4840	2.7694 4417	3.0642 9881	3.3901 2307	4.1478 4260
83	2.5308 0319	2.8040 6222	3.1064 3291	3.4409 7492	4.2204 2984
84	2.5592 7473	2.8391 1300	3.1491 4637	3.4925 8954	4.2942 8737
85	2.5880 6657	2.8746 0191	3.1924 4713	3.5449 7838	4.3694 3740
86	2.6171 8232	2.9105 3444	3.2363 4328	3.5981 5306	4.4459 0255
87	2.6466 2562	2.9469 1612	3.2808 4300	3.6521 2535	4.5237 0581
88	2.6764 0016	2.9837 5257	3.3259 5459	3.7069 0723	4.6028 7070
89	2.7065 0966	3.0210 4948	3.3716 8646	3.7625 1084	4.6834 2093
90	2.7369 5789	3.0588 1260	3.4180 4715	3.8189 4851	4.7653 8080
91	2.7677 4867	3.0970 4775	3.4650 4530	3.8762 3273	4.8487 7496
92	2.7988 8584	3.1357 6085	3.5126 8967	3.9343 7622	4.9336 2853
93	2.8303 7331	3.1749 5786	3.5609 8916	3.9933 9187	5.0199 6703
94	2.8622 1501	3.2146 4483	3.6099 5276	4.0532 9275	5.1078 1645
95	2.8944 1492	3.2548 2789	3.6595 8961	4.1140 9214	5.1972 0324
96	2.9269 7709	3.2955 1324	3.7099 0897	4.1758 0352	5.2881 5429
97	2.9599 0559	3.3367 0716	3.7609 2021	4.2384 4057	5.3806 9699
98	2.9932 0452	3.3784 1600	3.8126 3287	4.3020 1718	5.4748 5919
99	3.0268 7807	3.4206 4620	3.8650 5657	4.3665 4744	5.5706 6923
100	3.0609 3045	3.4634 0427	3.9182 0110	4.4320 4565	5.6681 5594

TABLE III. The Amount of 1 at Compound Interest

$(1+i)^n$

n	2%	$2\frac{1}{4}$%	$2\frac{1}{2}$%	$2\frac{3}{4}$%	3%
1	1.0200 0000	1.0225 0000	1.0250 0000	1.0275 0000	1.0300 0000
2	1.0404 0000	1.0455 0625	1.0506 2500	1.0557 5625	1.0609 0000
3	1.0612 0800	1.0690 3014	1.0768 9063	1.0847 8955	1.0927 2700
4	1.0824 3216	1.0930 8332	1.1038 1289	1.1146 2126	1.1255 0881
5	1.1040 8080	1.1176 7769	1.1314 0821	1.1452 7334	1.1592 7407
6	1.1261 6242	1.1428 2544	1.1596 9342	1.1767 6836	1.1940 5230
7	1.1486 8567	1.1685 3901	1.1886 8575	1.2091 2949	1.2298 7387
8	1.1716 5938	1.1948 3114	1.2184 0290	1.2423 8055	1.2667 8007
9	1.1950 9257	1.2217 1484	1.2488 6297	1.2765 4602	1.3047 7318
10	1.2189 9442	1.2492 0343	1.2800 8454	1.3116 5103	1.3439 1638
11	1.2433 7431	1.2773 1050	1.3120 8666	1.3477 2144	1.3842 3387
12	1.2682 4179	1.3060 4999	1.3448 8882	1.3847 8378	1.4257 6089
13	1.2936 0663	1.3354 3611	1.3785 1104	1.4228 6533	1.4685 3371
14	1.3194 7876	1.3654 8343	1.4129 7382	1.4619 9413	1.5125 8972
15	1.3458 6834	1.3962 0680	1.4482 9817	1.5021 9896	1.5579 6742
16	1.3727 8571	1.4276 2146	1.4845 0562	1.5435 0944	1.6047 0644
17	1.4002 4142	1.4597 4294	1.5216 1826	1.5859 5595	1.6528 4763
18	1.4282 4625	1.4925 8716	1.5596 5872	1.6295 6973	1.7024 3306
19	1.4568 1117	1.5261 7037	1.5986 5019	1.6743 8290	1.7535 0605
20	1.4859 4740	1.5605 0920	1.6386 1644	1.7204 2843	1.8061 1123
21	1.5156 6634	1.5956 2066	1.6795 8185	1.7677 4021	1.8602 9457
22	1.5459 7967	1.6315 2212	1.7215 7140	1.8163 5307	1.9161 0341
23	1.5768 9926	1.6682 3137	1.7646 1068	1.8663 0278	1.9735 8651
24	1.6084 3725	1.7057 6658	1.8087 2595	1.9176 2610	2.0327 9411
25	1.6406 0599	1.7441 4632	1.8539 4410	1.9703 6082	2.0937 7793
26	1.6734 1811	1.7833 8962	1.9002 9270	2.0245 4575	2.1565 9127
27	1.7068 8648	1.8235 1588	1.9478 0002	2.0802 2075	2.2212 8901
28	1.7410 2421	1.8645 4499	1.9964 9502	2.1[illegible]74 2682	2.2879 2768
29	1.7758 4469	1.9064 9725	2.0464 0737	2.1962 0606	2.3565 6551
30	1.8113 6158	1.9493 9344	2.0975 6758	2.2566 0173	2.4272 6247
31	1.8475 8882	1.9932 5479	2.1500 0677	2.3186 5828	2.5000 8035
32	1.8845 4059	2.0381 0303	2.2037 5694	2.3824 2138	2.5750 8276
33	1.9222 3140	2.0839 6034	2.2588 5086	2.4479 3797	2.6523 3524
34	1.9606 7603	2.1308 4945	2.3153 2213	2.5152 5626	2.7319 0530
35	1.9998 8955	2.1787 9356	2.3732 0519	2.5844 2581	2.8138 6245
36	2.0398 8734	2.2278 1642	2.4325 3532	2.6554 9752	2.8982 7833
37	2.0806 8509	2.2779 4229	2.4933 4870	2.7285 2370	2.9852 2668
38	2.1222 9879	2.3291 9599	2.5556 8242	2.8035 5810	3.0747 8348
39	2.1647 4477	2.3816 0290	2.6195 7448	2.8806 5595	3.1670 2608
40	2.2080 3966	2.4351 8897	2.6850 6384	2.9598 7399	3.2620 3779
41	2.2522 0046	2.4899 8072	2.7521 9043	3.0412 7052	3.3598 9893
42	2.2972 4447	2.5460 0528	2.8209 9520	3.1249 0546	3.4606 9589
43	2.3431 8936	2.6032 9040	2.8915 2008	3.2108 4036	3.5645 1677
44	2.3900 5314	2.6618 6444	2.9638 0808	3.2991 3847	3.6714 5227
45	2.4378 5421	2.7217 5639	3.0379 0328	3.3898 6478	3.7815 9584
46	2.4866 1129	2.7829 9590	3.1138 5086	3.4830 8606	3.8950 4372
47	2.5363 4351	2.8456 1331	3.1916 9713	3.5788 7093	4.0118 9503
48	2.5870 7039	2.9096 3961	3.2714 8956	3.6772 8988	4.1322 5188
49	2.6388 1179	2.9751 0650	3.3532 7680	3.7784 1535	4.2562 1944
50	2.6915 8803	3.0420 4640	3.4371 0872	3.8823 2177	4.3839 0602

TABLE III. The Amount of 1 at Compound Interest

$$(1+i)^n$$

n	2%	$2\frac{1}{4}$%	$2\frac{1}{2}$%	$2\frac{3}{4}$%	3%
51	2.7454 1979	3.1104 9244	3.5230 3644	3.9890 8562	4.5154 2320
52	2.8003 2819	3.1804 7852	3.6111 1235	4.0987 8547	4.6508 8590
53	2.8563 3475	3.2520 3929	3.7013 9016	4.2115 0208	4.7904 1247
54	2.9134 6144	3.3252 1017	3.7939 2491	4.3273 1838	4.9341 2485
55	2.9717 3067	3.4000 2740	3.8887 7303	4.4463 1964	5.0821 4859
56	3.0311 6529	3.4765 2802	3.9859 9236	4.5685 9343	5.2346 1305
57	3.0917 8859	3.5547 4990	4.0856 4217	4.6942 2975	5.3916 5144
58	3.1536 2436	3.6347 3177	4.1877 8322	4.8233 2107	5.5534 0098
59	3.2166 9685	3.7165 1324	4.2924 7780	4.9559 6239	5.7200 0301
60	3.2810 3079	3.8001 3479	4.3997 8975	5.0922 5136	5.8916 0310
61	3.3466 5140	3.8856 3782	4.5097 8449	5.2322 8827	6.0683 5120
62	3.4135 8443	3.9730 6467	4.6225 2910	5.3761 7620	6.2504 0173
63	3.4818 5612	4.0624 5862	4.7380 9233	5.5240 2105	6.4379 1379
64	3.5514 9324	4.1538 6394	4.8565 4464	5.6759 3162	6.6310 5120
65	3.6225 2311	4.2473 2588	4.9779 5826	5.8320 1974	6.8299 8273
66	3.6949 7357	4.3428 9071	5.1024 0721	5.9924 0029	7.0348 8222
67	3.7688 7304	4.4406 0576	5.2299 6739	6.1571 9130	7.2459 2868
68	3.8442 5050	4.5405 1939	5.3607 1658	6.3265 1406	7.4633 0654
69	3.9211 3551	4.6426 8107	5.4947 3449	6.5004 9319	7.6872 0574
70	3.9995 5822	4.7471 4140	5.6321 0286	6.6792 5676	7.9178 2191
71	4.0795 4939	4.8539 5208	5.7729 0543	6.8629 3632	8.1553 5657
72	4.1611 4038	4.9631 6600	5.9172 2806	7.0516 6706	8.4000 1727
73	4.2443 6318	5.0748 3723	6.0651 5876	7.2455 8791	8.6520 1778
74	4.3292 5045	5.1890 2107	6.2167 8773	7.4448 4158	8.9115 7832
75	4.4158 3546	5.3057 7405	6.3722 0743	7.6495 7472	9.1789 2567
76	4.5041 5216	5.4251 5396	6.5315 1261	7.8599 3802	9.4542 9344
77	4.5942 3521	5.5472 1993	6.6948 0043	8.0760 8632	9.7379 2224
78	4.6861 1991	5.6720 3237	6.8621 7044	8.2981 7869	10.0300 5991
79	4.7798 4231	5.7996 5310	7.0337 2470	8.5263 7861	10.3309 6171
80	4.8754 3916	5.9301 4530	7.2095 6782	8.7608 5402	10.6408 9056
81	4.9729 4794	6.0635 7357	7.3898 0701	9.0017 7751	10.9601 1727
82	5.0724 0690	6.2000 0397	7.5745 5219	9.2493 2639	11.2889 2079
83	5.1738 5504	6.3395 0406	7.7639 1599	9.5036 8286	11.6275 8842
84	5.2773 3214	6.4821 4290	7.9580 1389	9.7650 3414	11.9764 1607
85	5.3828 7878	6.6279 9112	8.1569 6424	10.0335 7258	12.3357 0855
86	5.4905 3636	6.7771 2092	8.3608 8834	10.3094 9583	12.7057 7981
87	5.6003 4708	6.9296 0614	8.5699 1055	10.5930 0696	13.0869 5320
88	5.7123 5402	7.0855 2228	8.7841 5832	10.8843 1465	13.4795 6180
89	5.8266 0110	7.2449 4653	9.0037 6228	11.1836 3331	13.8839 4865
90	5.9431 3313	7.4079 5782	9.2288 5633	11.4911 8322	14.3004 6711
91	6.0619 9579	7.5746 3688	9.4595 7774	11.8071 9076	14.7294 8112
92	6.1832 3570	7.7450 6621	9.6960 6718	12.1318 8851	15.1713 6556
93	6.3069 0042	7.9193 3020	9.9384 6886	12.4655 1544	15.6265 0652
94	6.4330 3843	8.0975 1512	10.1869 3058	12.8083 1711	16.0953 0172
95	6.5616 9920	8.2797 0921	10.4416 0385	13.1605 4584	16.5781 6077
96	6.6929 3318	8.4660 0267	10.7026 4395	13.5224 6085	17.0755 0559
97	6.8267 9184	8.6564 8773	10.9702 1004	13.8943 2852	17.5877 7076
98	6.9633 2768	8.8512 5871	11.2444 6530	14.2764 2255	18.1154 0388
99	7.1025 9423	9.0504 1203	11.5255 7693	14.6690 2417	18.6588 6600
100	7.2446 4612	9.2540 4630	11.8137 1635	15.0724 2234	19.2186 3198

TABLE III. The Amount of 1 at Compound Interest

$$(1+i)^n$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$
1	1.0350 0000	1.0400 0000	1.0450 0000	1.0500 0000	1.0550 0000
2	1.0712 2500	1.0816 0000	1.0920 2500	1.1025 0000	1.1130 2500
3	1.1087 1788	1.1248 6400	1.1411 6613	1.1576 2500	1.1742 4138
4	1.1475 2300	1.1698 5856	1.1925 1860	1.2155 0625	1.2388 2465
5	1.1876 8631	1.2166 5290	1.2461 8194	1.2762 8156	1.3069 6001
6	1.2292 5533	1.2653 1902	1.3022 6012	1.3400 9564	1.3788 4281
7	1.2722 7926	1.3159 3178	1.3608 6183	1.4071 0042	1.4546 7916
8	1.3168 0904	1.3685 6905	1.4221 0061	1.4774 5544	1.5346 8651
9	1.3628 9735	1.4233 1181	1.4860 9514	1.5513 2822	1.6190 9427
10	1.4105 9876	1.4802 4428	1.5529 6942	1.6288 9463	1.7081 4446
11	1.4599 6972	1.5394 5406	1.6228 5305	1.7103 3936	1.8020 9240
12	1.5110 6866	1.6010 3222	1.6958 8143	1.7958 5633	1.9012 0749
13	1.5639 5606	1.6650 7351	1.7721 9610	1.8856 4914	2.0057 7390
14	1.6186 9452	1.7316 7645	1.8519 4492	1.9799 3160	2.1160 9146
15	1.6753 4883	1.8009 4351	1.9352 8244	2.0789 2818	2.2324 7649
16	1.7339 8604	1.8729 8125	2.0223 7015	2.1828 7459	2.3552 6270
17	1.7946 7555	1.9479 0050	2.1133 7681	2.2920 1832	2.4848 0215
18	1.8574 8920	2.0258 1652	2.2084 7877	2.4066 1923	2.6214 6627
19	1.9225 0132	2.1068 4918	2.3078 6031	2.5269 5020	2.7656 4691
20	1.9897 8886	2.1911 2314	2.4117 1402	2.6532 9771	2.9177 5749
21	2.0594 3147	2.2787 6807	2.5202 4116	2.7859 6259	3.0782 3415
22	2.1315 1158	2.3699 1879	2.6336 5201	2.9252 6072	3.2475 3703
23	2.2061 1448	2.4647 1554	2.7521 6635	3.0715 2376	3.4261 5157
24	2.2833 2849	2.5633 0416	2.8760 1383	3.2250 9994	3.6145 8990
25	2.3632 4498	2.6658 3633	3.0054 3446	3.3863 5494	3.8133 9235
26	2.4459 5856	2.7724 6978	3.1406 7901	3.5556 7269	4.0231 2893
27	2.5315 6711	2.8833 6858	3.2820 0956	3.7334 5632	4.2444 0102
28	2.6201 7196	2.9987 0332	3.4296 9909	3.9201 2914	4.4778 4307
29	2.7118 7798	3.1186 5145	3.5840 3649	4.1161 3560	4.7241 2444
30	2.8067 9370	3.2433 9751	3.7453 1813	4.3219 4238	4.9839 5129
31	2.9050 3148	3.3731 3341	3.9138 5745	4.5380 3940	5.2580 6861
32	3.0067 0759	3.5080 5875	4.0839 8104	4.7649 4147	5.5472 6238
33	3.1119 4235	3.6483 8110	4.2740 3018	5.0031 8854	5.8523 6181
34	3.2208 6033	3.7943 1634	4.4663 6154	5.2533 4797	6.1742 4171
35	3.3335 9045	3.9460 8899	4.6673 4781	5.5160 1537	6.5138 2501
36	3.4502 6611	4.1039 3255	4.8773 7846	5.7918 1614	6.8720 8538
37	3.5710 2543	4.2680 8986	5.0968 6049	6.0814 0694	7.2500 5008
38	3.6960 1132	4.4388 1345	5.3262 1921	6.3854 7729	7.6488 0283
39	3.8253 7171	4.6163 6599	5.5658 9908	6.7047 5115	8.0694 8699
40	3.9592 5972	4.8010 2063	5.8163 6454	7.0399 8871	8.5133 0877
41	4.0978 3381	4.9930 6145	6.0781 0094	7.3919 8815	8.9815 4076
42	4.2412 5799	5.1927 8391	6.3516 1548	7.7615 8756	9.4755 2550
43	4.3897 0202	5.4004 9527	6.6374 3818	8.1496 6693	9.9966 7940
44	4.5433 4160	5.6165 1508	6.9361 2290	8.5571 5028	10.5464 9677
45	4.7023 5855	5.8411 7568	7.2482 4843	8.9850 0779	11.1265 5409
46	4.8669 4110	6.0748 2271	7.5744 1961	9.4342 5818	11.7385 1456
47	5.0372 8404	6.3178 1562	7.9152 6849	9.9059 7109	12.3841 3287
48	5.2135 8898	6.5705 2824	8.2714 5557	10.4012 6965	13.0652 6017
49	5.3960 5459	6.8333 4937	8.6436 7107	10.9213 3313	13.7838 4948
50	5.5849 2686	7.1066 8335	9.0326 3627	11.4673 9979	14.5419 6120

TABLE III. The Amount of 1 at Compound Interest

$(1+i)^n$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$
51	5.7803 9930	7.3909 5068	9.4391 0490	12.0407 6978	15.3417 6907
52	5.9827 1327	7.6865 8871	9.8638 6463	12.6428 0826	16.1855 6637
53	6.1921 0824	7.9940 5226	10.3077 3853	13.2749 4868	17.0757 7252
54	6.4088 3202	8.3138 1435	10.7715 8677	13.9386 9611	18.0149 4001
55	6.6331 4114	8.6463 6692	11.2563 0817	14.6356 3092	19.0057 6171
56	6.8653 0108	8.9922 2160	11.7628 4204	15.3674 1246	20.0510 7860
57	7.1055 8662	9.3519 1046	12.2921 6993	16.1357 8309	21.1538 8793
58	7.3542 8215	9.7259 8688	12.8453 1758	16.9425 7224	22.3173 5176
59	7.6116 8203	10.1150 2635	13.4233 5687	17.7897 0085	23.5448 0611
60	7.8780 9090	10.5196 2741	14.0274 0793	18.6791 8589	24.8397 7045
61	8.1538 2408	10.9404 1250	14.6586 4129	19.6131 4519	26.2059 5782
62	8.4392 0793	11.3780 2900	15.3182 8014	20.5938 0245	27.6472 8550
63	8.7345 8020	11.8331 5016	16.0076 0275	21.6234 9257	29.1678 8620
64	9.0402 9051	12.3064 7617	16.7279 4487	22.7046 6720	30.7721 1994
65	9.3567 0068	12.7987 3522	17.4807 0239	23.8399 0056	32.4645 8654
66	9.6841 8520	13.3106 8463	18.2673 3400	25.0318 9559	34.2501 3880
67	10.0231 3168	13.8431 1201	19.0893 6403	26.2834 9037	36.1338 9643
68	10.3739 4129	14.3968 3649	19.9483 8541	27.5976 6488	38.1212 6074
69	10.7370 2924	14.9727 0995	20.8460 6276	28.9775 4813	40.2179 3008
70	11.1128 2526	15.5716 1835	21.7841 3558	30.4264 2554	42.4299 1623
71	11.5017 7414	16.1944 8308	22.7644 2168	31.9477 4681	44.7635 6163
72	11.9043 3624	16.8422 6241	23.7888 2066	33.5451 3415	47.2255 5751
73	12.3209 8801	17.5159 5290	24.8593 1759	35.2223 9086	49.8229 6318
74	12.7522 2259	18.2165 9102	25.9779 8688	36.9835 1040	52.5632 2615
75	13.1985 5038	18.9452 5466	27.1469 9629	38.8326 8592	55.4542 0359
76	13.6604 9964	19.7030 6485	28.3686 1112	40.7743 2022	58.5041 8479
77	14.1386 1713	20.4911 8744	29.6451 9862	42.8130 3623	61.7219 1495
78	14.6334 6873	21.3108 3494	30.9792 3256	44.9536 8804	65.1166 2027
79	15.1456 4013	22.1632 6834	32.3732 9802	47.2013 7244	68.6980 3439
80	15.6757 3754	23.0497 9907	33.8300 9643	49.5614 4107	72.4764 2628
81	16.2243 8835	23.9717 9103	35.3524 5077	52.0395 1312	76.4626 2973
82	16.7922 4195	24.9306 6267	36.9433 1106	54.6414 8878	80.6680 7436
83	17.3799 7041	25.9278 8918	38.6057 6006	57.3735 6322	85.1048 1845
84	17.9882 6938	26.9650 0475	40.3430 1926	60.2422 4138	89.7855 8347
85	18.6178 5881	28.0436 0494	42.1584 5513	63.2543 5344	94.7237 9056
86	19.2694 8387	29.1653 4914	44.0555 8561	66.4170 7112	99.9335 9904
87	19.9439 1580	30.3319 6310	46.0380 8696	69.7379 2467	105.4299 4698
88	20.6419 5285	31.5452 4163	48.1098 0087	73.2248 2091	111.2285 9407
89	21.3644 2120	32.8070 5129	50.2747 4191	76.8860 6195	117.3461 6674
90	22.1121 7595	34.1193 3334	52.5371 0530	80.7303 6505	123.8002 0591
91	22.8861 0210	35.4841 0668	54.9012 7503	84.7668 8330	130.6092 1724
92	23.6871 1568	36.9034 7094	57.3718 3241	89.0052 2747	137.7927 2419
93	24.5161 6473	38.3796 0978	59.9535 6487	93.4554 8884	145.3713 2402
94	25.3742 3049	39.9147 9417	62.6514 7529	98.1282 6328	153.3667 4684
95	26.2623 2856	41.5113 8594	65.4707 9168	103.0346 7645	161.8019 1791
96	27.1815 1006	43.1718 4138	68.4169 7730	108.1864 1027	170.7010 2340
97	28.1328 6291	44.8987 1503	71.4957 4128	113.5957 3078	180.0895 7969
98	29.1175 1311	46.6946 6363	74.7130 4964	119.2755 1732	189.9945 0657
99	30.1366 2607	48.5624 5018	78.0751 3687	125.2392 9319	200.4442 0443
100	31.1914 0798	50.5049 4818	81.5885 1803	131.5012 5785	211.4686 3567

TABLE III. The Amount of 1 at Compound Interest

$$(1+i)^n$$

n	6%	$6\frac{1}{2}$%	7%	$7\frac{1}{2}$%	8%
1	1.0600 0000	1.0650 0000	1.0700 0000	1.0750 0000	1.0800 0000
2	1.1236 0000	1.1342 2500	1.1449 0000	1.1556 2500	1.1664 0000
3	1.1910 1600	1.2079 4963	1.2250 4300	1.2422 9688	1.2597 1200
4	1.2624 7696	1.2864 6635	1.3107 9601	1.3354 6914	1.3604 8896
5	1.3382 2558	1.3700 8666	1.4025 5173	1.4356 2933	1.4693 2808
6	1.4185 1911	1.4591 4230	1.5007 3035	1.5433 0153	1.5868 7432
7	1.5036 3026	1.5539 8655	1.6057 8148	1.6590 4914	1.7138 2427
8	1.5938 4807	1.6549 9567	1.7181 8618	1.7834 7783	1.8509 3021
9	1.6894 7896	1.7625 7039	1.8384 5921	1.9172 3866	1.9990 0463
10	1.7908 4770	1.8771 3747	1.9671 5136	2.0610 3156	2.1589 2500
11	1.8982 9856	1.9991 5140	2.1048 5195	2.2156 0893	2.3316 3900
12	2.0121 9647	2.1290 9624	2.2521 9159	2.3817 7960	2.5181 7012
13	2.1329 2826	2.2674 8750	2.4098 4500	2.5604 1307	2.7196 2373
14	2.2609 0396	2.4148 7418	2.5785 3415	2.7524 4405	2.9371 9362
15	2.3965 5819	2.5718 4102	2.7590 3154	2.9588 7735	3.1721 6911
16	2.5403 5168	2.7390 1067	2.9521 6375	3.1807 9315	3.4259 4264
17	2.6927 7279	2.9170 4637	3.1588 1521	3.4193 5264	3.7000 1805
18	2.8543 3915	3.1066 5438	3.3799 3228	3.6758 0409	3.9960 1950
19	3.0255 9950	3.3085 8691	3.6165 2754	3.9514 8940	4.3157 0106
20	3.2071 3547	3.5236 4506	3.8696 8446	4.2478 5110	4.6609 5714
21	3.3995 6360	3.7526 8199	4.1405 6237	4.5664 3993	5.0338 3372
22	3.6035 3742	3.9966 0632	4.4304 0174	4.9089 2293	5.4365 4041
23	3.8197 4066	4.2563 8573	4.7405 2986	5.2770 9215	5.8714 6365
24	4.0489 3464	4.5330 5081	5.0723 6695	5.6728 7406	6.3411 8074
25	4.2918 7072	4.8276 9911	5.4274 3264	6.0983 3961	6.8484 7520
26	4.5493 8296	5.1414 9955	5.8073 5292	6.5557 1508	7.3963 5321
27	4.8223 4594	5.4756 9702	6.2138 6763	7.0473 9371	7.9880 6147
28	5.1116 8670	5.8316 1733	6.6488 3836	7.5759 4824	8.6271 0639
29	5.4183 8790	6.2106 7245	7.1142 5705	8.1441 4436	9.3172 7490
30	5.7434 9117	6.6143 6616	7.6122 5504	8.7549 5519	10.0626 5689
31	6.0881 0064	7.0442 9996	8.1451 1290	9.4115 7683	10.8676 6944
32	6.4533 8668	7.5021 7946	8.7152 7080	10.1174 4509	11.7370 8300
33	6.8405 8988	7.9898 2113	9.3253 3975	10.8762 5347	12.6760 4964
34	7.2510 2528	8.5091 5950	9.9781 1354	11.6919 7248	13.6901 3361
35	7.6860 8679	9.0622 5487	10.6765 8148	12.5688 7042	14.7853 4429
36	8.1472 5200	9.6513 0143	11.4239 4219	13.5115 3570	15.9681 7184
37	8.6360 8712	10.2786 3603	12.2236 1814	14.5249 0088	17.2456 2558
38	9.1542 5235	10.9467 4737	13.0792 7141	15.6142 6844	18.6252 7563
39	9.7035 0749	11.6582 8595	13.9948 2041	16.7853 3858	20.1152 9768
40	10.2857 1794	12.4160 7453	14.9744 5784	18.0442 3897	21.7245 2150
41	10.9028 6101	13.2231 1938	16.0226 6989	19.3975 5689	23.4624 8322
42	11.5570 3267	14.0826 2214	17.1442 5678	20.8523 7366	25.3394 8187
43	12.2504 5463	14.9979 9258	18.3443 5475	22.4163 0168	27.3666 4042
44	12.9854 8191	15.9728 6209	19.6284 5959	24.0975 2431	29.5559 7166
45	13.7646 1083	17.0110 9813	21.0024 5176	25.9048 3863	31.9204 4939
46	14.5904 8748	18.1168 1951	22.4726 2338	27.8477 0153	34.4740 8534
47	15.4659 1673	19.2944 1278	24.0457 0702	29.9362 7915	37.2320 1217
48	16.3938 7173	20.5485 4961	25.7289 0651	32.1815 0008	40.2105 7314
49	17.3775 0403	21.8842 0533	27.5299 2997	34.5951 1259	43.4274 1899
50	18.4201 5427	23.3066 7868	29.4570 2506	37.1897 4603	46.9016 1251

TABLE IV. The Present Value of 1 at Compound Interest

$$v^n = (1+i)^{-n}$$

n	$\frac{1}{8}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{8}\%$	1%
1	0.9966 7774	0.9958 5062	0.9950 2488	0.9913 2590	0.9900 9901
2	0.9933 6652	0.9917 1846	0.9900 7450	0.9827 2704	0.9802 9605
3	0.9900 6630	0.9876 0345	0.9851 4876	0.9742 0276	0.9705 9015
4	0.9867 7704	0.9835 0551	0.9802 4752	0.9657 5243	0.9609 8034
5	0.9834 9871	0.9794 2457	0.9753 7067	0.9573 7539	0.9514 6569
6	0.9802 3127	0.9753 6057	0.9705 1808	0.9490 7102	0.9420 4524
7	0.9769 7469	0.9713 1343	0.9656 8963	0.9408 3868	0.9327 1805
8	0.9737 2893	0.9672 8308	0.9608 8520	0.9326 7775	0.9234 8322
9	0.9794 9395	0.9632 6946	0.9561 0468	0.9245 8761	0.9143 3982
10	0.9672 6972	0.9592 7249	0.9513 4794	0.9165 6765	0.9052 8695
11	0.9640 5620	0.9552 9211	0.9466 1489	0.9086 1724	0.8963 2372
12	0.9608 5335	0.9513 2824	0.9419 0534	0.9007 3581	0.8874 4923
13	0.9576 6115	0.9473 8082	0.9372 1924	0.9029 2273	0.8786 6260
14	0.9544 7955	0.9434 4978	0.9325 5646	0.8851 7743	0.8699 6297
15	0.9513 0852	0.9395 3505	0.9279 1688	0.8774 9931	0.8613 4947
16	0.9481 4803	0.9356 3656	0.9233 0037	0.8698 8779	0.8528 2126
17	0.9449 9803	0.9317 5425	0.9187 0684	0.8623 4230	0.8443 7749
18	0.9418 5851	0.9278 8805	0.9141 3616	0.8548 6225	0.8360 1731
19	0.9387 2941	0.9240 3789	0.9095 8822	0.8474 4709	0.8277 3992
20	0.9356 1071	0.9202 0371	0.9050 6290	0.8400 9624	0.8195 4447
21	0.9325 0236	0.9163 8544	0.9005 6010	0.8328 0917	0.8114 3017
22	0.9294 0435	0.9125 8301	0.8960 7971	0.8255 8530	0.8033 9621
23	0.9263 1663	0.9087 9636	0.8916 2160	0.8184 2409	0.7954 4179
24	0.9232 3916	0.9050 2542	0.8871 8567	0.8113 2499	0.7875 6613
25	0.9201 7192	0.9012 7012	0.8827 7181	0.8042 8748	0.7797 6844
26	0.9171 1487	0.8975 3041	0.8783 7991	0.7973 1101	0.7720 4796
27	0.9140 6798	0.8938 0622	0.8740 0986	0.7903 9505	0.7644 0392
28	0.9110 3121	0.8900 9748	0.8696 6155	0.7835 3908	0.7568 3557
29	0.9080 0453	0.8864 0413	0.8653 3488	0.7767 4258	0.7493 4215
30	0.9049 8790	0.8827 2610	0.8610 2973	0.7700 0504	0.7419 2292
31	0.9019 8130	0.8790 6334	0.8567 4600	0.7633 2594	0.7345 7715
32	0.8989 8468	0.8754 1577	0.8524 8358	0.7567 0477	0.7273 0411
33	0.8959 9802	0.8717 8334	0.8482 4237	0.7501 4104	0.7201 0307
34	0.8930 2128	0.8681 6599	0.8440 2226	0.7436 3424	0.7129 7334
35	0.8900 5444	0.8645 6364	0.8398 2314	0.7371 8388	0.7059 1420
36	0.8870 9745	0.8609 7624	0.8356 4492	0.7307 9847	0.6989 2495
37	0.8841 5028	0.8574 0372	0.8314 8748	0.7244 5053	0.6920 0490
38	0.8812 1290	0.8538 4603	0.8273 5073	0.7181 6657	0.6851 5337
39	0.8782 8528	0.8503 0310	0.8232 3455	0.7119 3712	0.6783 6967
40	0.8753 6739	0.8467 7487	0.8191 3886	0.7057 6171	0.6716 5314
41	0.8724 5920	0.8432 6128	0.8150 6354	0.6996 3986	0.6650 0311
42	0.8695 6066	0.8397 6227	0.8110 0850	0.6935 7111	0.6584 1892
43	0.8666 7175	0.8362 7778	0.8069 7363	0.6875 5500	0.6518 9992
44	0.8637 9245	0.8328 0775	0.8029 5884	0.6815 9108	0.6454 4546
45	0.8609 2270	0.8293 5211	0.7989 6402	0.6756 7889	0.6390 5492
46	0.8580 6249	0.8259 1082	0.7949 8907	0.6698 1798	0.6327 2764
47	0.8552 1179	0.8224 8380	0.7910 3390	0.6640 0792	0.6264 6301
48	0.8523 7055	0.8190 7100	0.7870 9841	0.6582 4824	0.6202 6041
49	0.8495 3876	0.8156 7237	0.7831 8250	0.6525 3853	0.6141 1921
50	0.8467 1637	0.8122 8784	0.7792 8607	0.6468 7835	0.6080 3882

TABLE IV. The Present Value of 1 at Compound Interest

$$v^n = (1 + i)^{-n}$$

n	$\frac{1}{8}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{8}\%$	1%
51	0.8439 0336	0.8089 1735	0.7754 0902	0.6412 6726	0.6020 1864
52	0.8410 9969	0.8055 6084	0.7715 5127	0.6357 0484	0.5960 5806
53	0.8383 0534	0.8022 1827	0.7677 1270	0.6301 9067	0.5901 5649
54	0.8355 2027	0.7988 8956	0.7638 9324	0.6247 2433	0.5843 1336
55	0.8327 4446	0.7955 7467	0.7600 9277	0.6193 0541	0.5785 2808
56	0.8299 7787	0.7922 7353	0.7563 1122	0.6139 3349	0.5728 0008
57	0.8272 2047	0.7889 8608	0.7525 4847	0.6086 0817	0.5671 2879
58	0.8244 7222	0.7857 1228	0.7488 0445	0.6033 2904	0.5615 1365
59	0.8217 3311	0.7824 5207	0.7450 7906	0.5980 9571	0.5559 5411
60	0.8190 0310	0.7792 0538	0.7413 7220	0.5929 0776	0.5504 4962
61	0.8162 8216	0.7759 7216	0.7376 8378	0.5877 6482	0.5449 9962
62	0.8135 7026	0.7727 5236	0.7340 1371	0.5826 6649	0.5396 0358
63	0.8108 6737	0.7695 4591	0.7303 6190	0.5776 1238	0.5342 6097
64	0.8081 7346	0.7663 5278	0.7267 2826	0.5726 0211	0.5289 7126
65	0.8054 8850	0.7631 7289	0.7231 1269	0.5676 3530	0.5237 3392
66	0.8028 1246	0.7600 0620	0.7195 1512	0.5627 1158	0.5185 4844
67	0.8001 4531	0.7568 5265	0.7159 3544	0.5578 3056	0.5134 1429
68	0.7974 8702	0.7537 1218	0.7123 7357	0.5529 9188	0.5083 3099
69	0.7948 3756	0.7505 8474	0.7088 2943	0.5481 9517	0.5032 9831
70	0.7921 9690	0.7474 7028	0.7053 0291	0.5434 4007	0.4983 1486
71	0.7895 6502	0.7443 6874	0.7017 9394	0.5387 2622	0.4933 8105
72	0.7869 4188	0.7412 8008	0.6983 0243	0.5340 5325	0.4884 9609
73	0.7843 2745	0.7382 0423	0.6948 2829	0.5294 2082	0.4836 5940
74	0.7817 2171	0.7351 4114	0.6913 7143	0.5248 2857	0.4788 7078
75	0.7791 2463	0.7320 9076	0.6879 3177	0.5202 7615	0.4741 2949
76	0.7765 3618	0.7290 5304	0.6845 0923	0.5157 6322	0.4694 3514
77	0.7739 5632	0.7260 2792	0.6811 0371	0.5112 8944	0.4647 8726
78	0.7713 8504	0.7230 1536	0.6777 1513	0.5068 5447	0.4601 8541
79	0.7688 2230	0.7200 1529	0.6743 4342	0.5024 5796	0.4556 2912
80	0.7662 6807	0.7170 2768	0.6709 8847	0.4980 9959	0.4511 1794
81	0.7637 2233	0.7140 5246	0.6676 5022	0.4937 7902	0.4466 5142
82	0.7611 8505	0.7110 8959	0.6643 2858	0.4894 9593	0.4422 2913
83	0.7586 5619	0.7081 3901	0.6610 2346	0.4852 4999	0.4378 5063
84	0.7561 3574	0.7052 0067	0.6577 3479	0.4810 4089	0.4335 1547
85	0.7536 2366	0.7022 7453	0.6544 6248	0.4768 6829	0.4292 2324
86	0.7511 1993	0.6993 6052	0.6512 0644	0.4727 3188	0.4249 7350
87	0.7486 2451	0.6964 5861	0.6479 6661	0.4686 3136	0.4207 6585
88	0.7461 3739	0.6935 6874	0.6447 4290	0.4645 6640	0.4165 9985
89	0.7436 5853	0.6906 9086	0.6415 3522	0.4605 3671	0.4124 7510
90	0.7411 8790	0.6878 2493	0.6383 4350	0.4565 4197	0.4083 9119
91	0.7387 2548	0.6849 7088	0.6351 6766	0.4525 8187	0.4043 4771
92	0.7362 7125	0.6821 2868	0.6320 0763	0.4486 5613	0.4003 4427
93	0.7338 2516	0.6792 9827	0.6288 6331	0.4447 6444	0.3963 8046
94	0.7313 8720	0.6764 7960	0.6257 3464	0.4409 0651	0.3924 5590
95	0.7389 5735	0.6736 7263	0.6226 2153	0.4370 8204	0.3885 7020
96	0.7265 3556	0.6708 7731	0.6195 2391	0.4332 9075	0.3847 2297
97	0.7241 2182	0.6680 9359	0.6164 4170	0.4295 3234	0.3809 1383
98	0.7217 1610	0.6653 2141	0.6133 7483	0.4258 0654	0.3771 4241
99	0.7193 1837	0.6625 6074	0.6103 2321	0.4221 1305	0.3734 0832
100	0.7169 2861	0.6598 1153	0.6072 8678	0.4184 5159	0.3697 1121

TABLE IV. The Present Value of 1 at Compound Interest

$$v = (1 + i)^{-n}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
1	0.9888 7515	0.9876 5432	0.9864 3650	0.9852 2167	0.9828 0098
2	0.9778 7407	0.9754 6106	0.9730 5696	0.9706 6175	0.9658 9777
3	0.9669 9537	0.9634 1833	0.9598 5890	0.9563 1699	0.9492 8528
4	0.9562 3770	0.9515 2428	0.9468 3986	0.9421 8423	0.9329 5851
5	0.9455 9970	0.9397 7706	0.9339 9739	0.9282 6033	0.9169 1254
6	0.9350 8005	0.9281 7488	0.9213 2912	0.9145 4219	0.9011 4254
7	0.9246 7743	0.9167 1593	0.9088 3267	0.9010 2679	0.8856 4378
8	0.9143 9054	0.9053 9845	0.8965 0571	0.8877 1112	0.8704 1157
9	0.9042 1808	0.8942 2069	0.8843 4596	0.8745 9224	0.8554 4135
10	0.8941 5881	0.8831 8093	0.8723 5113	0.8616 6723	0.8407 2860
11	0.8842 1142	0.8722 7746	0.8605 1809	0.8489 3323	0.8262 6889
12	0.8743 7470	0.8615 0860	0.8488 4734	0.8363 8742	0.8120 5788
13	0.8646 4742	0.8508 7260	0.8373 3400	0.8240 2702	0.7980 9128
14	0.8550 2835	0.8403 6800	0.8259 7682	0.8118 4928	0.7843 6490
15	0.8455 1629	0.8299 9318	0.8147 7368	0.7998 5150	0.7708 7459
16	0.8361 1005	0.8197 4635	0.8037 2250	0.7880 3104	0.7576 1631
17	0.8268 0846	0.8096 2602	0.7928 2120	0.7763 8526	0.7445 8605
18	0.8176 1034	0.7996 3064	0.7820 6777	0.7649 1159	0.7317 7990
19	0.8085 1455	0.7897 5866	0.7714 6020	0.7536 0747	0.7191 9401
20	0.7995 1995	0.7800 0855	0.7609 9649	0.7424 7042	0.7068 2458
21	0.7906 2542	0.7703 7881	0.7506 7472	0.7314 9795	0.6946 6789
22	0.7818 2983	0.7608 6796	0.7404 9294	0.7206 8763	0.6827 2028
23	0.7731 3210	0.7514 7453	0.7304 4926	0.7100 3708	0.6709 7817
24	0.7645 3112	0.7421 9707	0.7205 4181	0.6995 4392	0.6594 3800
25	0.7560 2583	0.7330 3414	0.7107 6874	0.6892 0583	0.6480 9632
26	0.7476 1516	0.7239 8434	0.7011 2823	0.6790 2052	0.6369 4970
27	0.7392 9806	0.7150 4626	0.6916 1847	0.6689 8574	0.6259 9479
28	0.7310 7348	0.7062 1853	0.6822 3771	0.6590 9925	0.6152 2829
29	0.7229 4040	0.6974 9978	0.6729 8417	0.6493 5887	0.6046 4697
30	0.7148 9780	0.6888 8867	0.6638 5615	0.6397 6243	0.5942 4764
31	0.7069 4467	0.6803 8387	0.6548 5194	0.6303 0781	0.5840 2716
32	0.6990 8002	0.6719 8407	0.6459 6985	0.6209 9292	0.5739 8247
33	0.6913 0287	0.6636 8797	0.6372 0824	0.6118 1568	0.5641 1053
34	0.6836 1223	0.6554 9429	0.6285 6546	0.6027 7407	0.5544 0839
35	0.6760 0715	0.6474 0177	0.6200 3991	0.5938 6608	0.5448 7311
36	0.6684 8667	0.6394 0916	0.6116 3000	0.5850 8974	0.5355 0183
37	0.6610 4986	0.6315 1522	0.6033 3416	0.5764 4309	0.5262 9172
38	0.6536 9578	0.6237 1873	0.5951 5083	0.5679 2423	0.5172 4002
39	0.6464 2352	0.6160 1850	0.5870 7850	0.5595 3126	0.5083 4400
40	0.6392 3216	0.6084 1334	0.5791 1566	0.5512 6232	0.4996 0098
41	0.6321 2080	0.6009 0206	0.5712 6083	0.5431 1559	0.4910 0834
42	0.6250 8855	0.5934 8352	0.5635 1253	0.5350 8925	0.4825 6348
43	0.6181 3454	0.5861 5656	0.5558 6933	0.5271 8153	0.4742 6386
44	0.6112 5789	0.5789 2006	0.5483 2979	0.5193 9067	0.4661 0699
45	0.6044 5774	0.5717 7290	0.5408 9252	0.5117 1494	0.4580 9040
46	0.5977 3324	0.5647 1397	0.5335 5612	0.5041 5265	0.4502 1170
47	0.5910 8355	0.5577 4219	0.5263 1923	0.4967 0212	0.4424 6850
48	0.5845 0784	0.5508 5649	0.5191 8050	0.4893 6170	0.4348 5848
49	0.5780 0528	0.5440 5579	0.5121 3860	0.4821 2975	0.4273 7934
50	0.5715 7506	0.5373 3905	0.5051 9220	0.4750 0468	0.4200 2883

TABLE IV. The Present Value of 1 at Compound Interest

$$v = (1 + i)^{-n}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
51	0.5652 1637	0.5307 0524	0.4983 4003	0.4679 8491	0.4128 0475
52	0.5589 2843	0.5241 5332	0.4915 8079	0.4610 6887	0.4057 0492
53	0.5527 1044	0.5176 8229	0.4849 1323	0.4542 5505	0.3987 2719
54	0.5465 6162	0.5112 9115	0.4783 3611	0.4475 4192	0.3918 6947
55	0.5404 8120	0.5049 7892	0.4718 4820	0.4409 2800	0.3851 2970
56	0.5344 6843	0.4987 4461	0.4654 4829	0.4344 1182	0.3785 0585
57	0.5285 2256	0.4925 8727	0.4591 3518	0.4279 9194	0.3719 9592
58	0.5226 4282	0.4865 0594	0.4529 0770	0.4216 6694	0.3655 9796
59	0.5168 2850	0.4804 9970	0.4467 6468	0.4154 3541	0.3593 1003
60	0.5110 7887	0.4745 6760	0.4407 0499	0.4092 9597	0.3531 3025
61	0.5053 9319	0.4687 0874	0.4347 2749	0.4032 4726	0.3470 5676
62	0.4997 7077	0.4629 2222	0.4288 3106	0.3972 8794	0.3410 8772
63	0.4942 1090	0.4572 0713	0.4230 1461	0.3914 1669	0.3352 2135
64	0.4887 1288	0.4515 6259	0.4172 7705	0.3856 3221	0.3294 5587
65	0.4832 7602	0.4459 8775	0.4116 1731	0.3799 3321	0.3237 8956
66	0.4778 9965	0.4404 8173	0.4060 3434	0.3743 1843	0.3182 2069
67	0.4725 8309	0.4350 4368	0.4005 2709	0.3687 8663	0.3127 4761
68	0.4673 2568	0.4296 7277	0.3950 9454	0.3633 3658	0.3073 6866
69	0.4621 2675	0.4243 6817	0.3897 3568	0.3579 6708	0.3020 8222
70	0.4569 8566	0.4191 2905	0.3844 4949	0.3526 7692	0.2968 8670
71	0.4519 0177	0.4139 5462	0.3792 3501	0.3474 6495	0.2917 8054
72	0.4468 7443	0.4088 4407	0.3740 9126	0.3423 3000	0.2867 6221
73	0.4419 0302	0.4037 9661	0.3690 1727	0.3372 7093	0.2818 3018
74	0.4369 8692	0.3988 1147	0.3640 1210	0.3322 8663	0.2769 8298
75	0.4321 2551	0.3938 8787	0.3590 7483	0.3273 7599	0.2722 1914
76	0.4273 1818	0.3890 2506	0.3542 0451	0.3225 3793	0.2675 3724
77	0.4225 6433	0.3842 2228	0.3494 0026	0.3177 7136	0.2629 3586
78	0.4178 6337	0.3794 7879	0.3446 6117	0.3130 7523	0.2584 1362
79	0.4132 1470	0.3747 9387	0.3399 8636	0.3084 4850	0.2539 6916
80	0.4086 1775	0.3701 6679	0.3353 7495	0.3038 9015	0.2496 0114
81	0.4040 7194	0.3655 9683	0.3308 2609	0.2993 9916	0.2453 0825
82	0.3995 7670	0.3610 8329	0.3263 3893	0.2949 7454	0.2410 8919
83	0.3951 3148	0.3566 2547	0.3219 1263	0.2906 1531	0.2369 4269
84	0.3907 3570	0.3522 2268	0.3175 4637	0.2863 2050	0.2328 6751
85	0.3863 8882	0.3478 7426	0.3132 3933	0.2820 8917	0.2288 6242
86	0.3820 9031	0.3435 7951	0.3089 9071	0.2779 2036	0.2249 2621
87	0.3778 3961	0.3393 3779	0.3047 9971	0.2738 1316	0.2210 5770
88	0.3736 3621	0.3351 4843	0.3006 6556	0.2697 6666	0.2172 5572
89	0.3694 7956	0.3310 1080	0.2965 8748	0.2657 7997	0.2135 1914
90	0.3653 6916	0.3269 2425	0.2925 6472	0.2618 5218	0.2098 4682
91	0.3613 0448	0.3228 8814	0.2885 9652	0.2579 8245	0.2062 3766
92	0.3572 8503	0.3189 0187	0.2846 8214	0.2541 6990	0.2026 9057
93	0.3533 1029	0.3149 6481	0.2808 2085	0.2504 1369	0.1992 0450
94	0.3493 7976	0.3110 7636	0.2770 1194	0.2467 1300	0.1957 7837
95	0.3454 9297	0.3072 3591	0.2732 5468	0.2430 6699	0.1924 1118
96	0.3416 4941	0.3034 4287	0.2695 4839	0.2394 7487	0.1891 0190
97	0.3378 4861	0.2996 9666	0.2658 9237	0.2359 3583	0.1858 4953
98	0.3340 9010	0.2959 9670	0.2622 8594	0.2324 4909	0.1826 5310
99	0.3303 7340	0.2923 4242	0.2587 2843	0.2290 1389	0.1795 1165
100	0.3266 9805	0.2887 3326	0.2552 1916	0.2256 2944	0.1764 2422

TABLE IV. The Present Value of 1 at Compound Interest

$$v^n = (1 + i)^{-n}$$

n	2%	$2\frac{1}{4}$%	$2\frac{1}{2}$%	$2\frac{3}{4}$%	3%
1	0.9803 9216	0.9779 9511	0.9756 0976	0.9732 3601	0.9708 7379
2	0.9611 6878	0.9564 7444	0.9518 1440	0.9471 8833	0.9425 9591
3	0.9423 2233	0.9354 2732	0.9285 9941	0.9218 3779	0.9151 4166
4	0.9238 4543	0.9148 4335	0.9059 5064	0.8971 6573	0.8884 8705
5	0.9057 3081	0.8947 1232	0.8838 5429	0.8731 5400	0.8626 0878
6	0.8879 7138	0.8750 2427	0.8622 9687	0.8497 8491	0.8374 8426
7	0.8705 6018	0.8557 6946	0.8412 6524	0.8270 4128	0.8130 9151
8	0.8534 9037	0.8369 3835	0.8207 4657	0.8049 0635	0.7894 0923
9	0.8367 5527	0.8185 2161	0.8007 2836	0.7833 6385	0.7664 1673
10	0.8203 4830	0.8005 1013	0.7811 9840	0.7623 9791	0.7440 9391
11	0.8042 6304	0.7828 9499	0.7621 4478	0.7419 9310	0.7224 2128
12	0.7884 9318	0.7656 6748	0.7435 5589	0.7221 3440	0.7013 7988
13	0.7730 3253	0.7488 1905	0.7254 2038	0.7028 0720	0.6809 5134
14	0.7578 7502	0.7323 4137	0.7077 2720	0.6839 9728	0.6611 1781
15	0.7430 1473	0.7162 2628	0.6904 6556	0.6656 9078	0.6418 6195
16	0.7284 4581	0.7004 6580	0.6736 2493	0.6478 7424	0.6231 6694
17	0.7141 6256	0.6850 5212	0.6571 9506	0.6305 3454	0.6050 1645
18	0.7001 5937	0.6699 7763	0.6411 6591	0.6136 5892	0.5873 9461
19	0.6864 3076	0.6552 3484	0.6255 2772	0.5972 3496	0.5702 8603
20	0.6729 7133	0.6408 1647	0.6102 7094	0.5812 5057	0.5536 7575
21	0.6597 7582	0.6267 1538	0.5953 8629	0.5656 9398	0.5375 4928
22	0.6468 3904	0.6129 2457	0.5808 6467	0.5505 5375	0.5218 9250
23	0.6341 5592	0.5994 3724	0.5666 9724	0.5358 1874	0.5066 9175
24	0.6217 2149	0.5862 4668	0.5528 7535	0.5214 7809	0.4919 3374
25	0.6095 3087	0.5733 4639	0.5393 9059	0.5075 2126	0.4776 0557
26	0.5975 7928	0.5607 2997	0.5262 3472	0.4939 3796	0.4636 9473
27	0.5858 6204	0.5483 9117	0.5133 9973	0.4807 1821	0.4501 8906
28	0.5743 7455	0.5363 2388	0.5008 7778	0.4678 5227	0.4370 7675
29	0.5631 1231	0.5245 2213	0.4886 6125	0.4553 3068	0.4243 4636
30	0.5520 7089	0.5129 8008	0.4767 4269	0.4431 4421	0.4119 8676
31	0.5412 4597	0.5016 9201	0.4651 1481	0.4312 8301	0.3999 8715
32	0.5306 3330	0.4906 5233	0.4537 7055	0.4197 4103	0.3883 3703
33	0.5202 2873	0.4798 5558	0.4427 0298	0.4085 0708	0.3770 2625
34	0.5100 2817	0.4692 9641	0.4319 0534	0.3975 7380	0.3660 4490
35	0.5000 2761	0.4589 6960	0.4213 7107	0.3869 3314	0.3553 8340
36	0.4902 2315	0.4488 7002	0.4110 9372	0.3765 7727	0.3450 3243
37	0.4806 1093	0.4389 9268	0.4010 6705	0.3664 9856	0.3349 8294
38	0.4711 8719	0.4293 3270	0.3912 8492	0.3566 8959	0.3252 2615
39	0.4619 4822	0.4198 8528	0.3817 4139	0.3471 4316	0.3157 5355
40	0.4528 9042	0.4106 4575	0.3724 3062	0.3378 5222	0.3065 5684
41	0.4440 1021	0.4016 0954	0.3633 4695	0.3288 0995	0.2976 2800
42	0.4353 0413	0.3927 7216	0.3544 8483	0.3200 0968	0.2889 5922
43	0.4267 6875	0.3841 2925	0.3458 3886	0.3114 4495	0.2805 4294
44	0.4184 0074	0.3756 7653	0.3374 0376	0.3031 0944	0.2723 7178
45	0.4101 9680	0.3674 0981	0.3291 7440	0.2949 9702	0.2644 3862
46	0.4021 5373	0.3593 2500	0.3211 4576	0.2871 0172	0.2567 3653
47	0.3942 6836	0.3514 1809	0.3133 1294	0.2794 1773	0.2492 5876
48	0.3865 3761	0.3436 8518	0.3056 7116	0.2719 3940	0.2419 9880
49	0.3789 5844	0.3361 2242	0.2982 1576	0.2646 6122	0.2349 5029
50	0.3715 2788	0.3287 2608	0.2909 4221	0.2575 7783	0.2281 0708

TABLE IV. The Present Value of 1 at Compound Interest

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$$v^n = (1+i)^{-n}$$

n	2%	$2\frac{1}{4}$%	$2\frac{1}{2}$%	$2\frac{3}{4}$%	3%
51	0.3642 4302	0.3214 9250	0.2838 4606	0.2506 8402	0.2214 6318
52	0.3571 0100	0.3144 1810	0.2769 2298	0.2439 7471	0.2150 1280
53	0.3500 9902	0.3074 9936	0.2701 6876	0.2374 4497	0.2087 5029
54	0.3432 3433	0.3007 3287	0.2635 7928	0.2310 9000	0.2026 7019
55	0.3365 0425	0.2941 1528	0.2571 5052	0.2249 0511	0.1967 6717
56	0.3299 0613	0.2876 4330	0.2508 7855	0.2188 8575	0.1910 3609
57	0.3234 3738	0.2813 1374	0.2447 5956	0.2130 2749	0.1854 7193
58	0.3170 9547	0.2751 2347	0.2387 8982	0.2073 2603	0.1800 6984
59	0.3108 7791	0.2690 6940	0.2329 6568	0.2017 7716	0.1748 2508
60	0.3047 8227	0.2631 4856	0.2272 8359	0.1963 7679	0.1697 3309
61	0.2988 0614	0.2573 5801	0.2217 4009	0.1911 2097	0.1647 8941
62	0.2929 4720	0.2516 9487	0.2163 3179	0.1860 0581	0.1599 8972
63	0.2872 0314	0.2461 5635	0.2110 5541	0.1810 2755	0.1553 2982
64	0.2815 7170	0.2407 3971	0.2059 0771	0.1761 8253	0.1508 0565
65	0.2760 5069	0.2354 4226	0.2008 8557	0.1714 6718	0.1464 1325
66	0.2706 3793	0.2302 6138	0.1959 8593	0.1668 7804	0.1421 4879
67	0.2653 3130	0.2251 9450	0.1912 0578	0.1624 1172	0.1380 0853
68	0.2601 2873	0.2202 3912	0.1865 4223	0.1580 6493	0.1339 8887
69	0.2550 2817	0.2153 9278	0.1819 9241	0.1538 3448	0.1300 8628
70	0.2500 2761	0.2106 5309	0.1775 5358	0.1497 1726	0.1262 9736
71	0.2451 2511	0.2060 1769	0.1732 2300	0.1457 1023	0.1226 1880
72	0.2403 1874	0.2014 8429	0.1689 9805	0.1418 1044	0.1190 4737
73	0.2356 0661	0.1970 5065	0.1648 7615	0.1380 1503	0.1155 7998
74	0.2309 8687	0.1927 1458	0.1608 5478	0.1343 2119	0.1122 1357
75	0.2264 5771	0.1884 7391	0.1569 3149	0.1307 2622	0.1089 4521
76	0.2220 1737	0.1843 2657	0.1531 0389	0.1272 2747	0.1057 7205
77	0.2176 6408	0.1802 7048	0.1493 6965	0.1238 2235	0.1026 9131
78	0.2133 9616	0.1763 0365	0.1457 2649	0.1205 0837	0.0997 0030
79	0.2092 1192	0.1724 2411	0.1421 7218	0.1172 8309	0.0967 9641
80	0.2051 0973	0.1686 2993	0.1387 0457	0.1141 4412	0.0939 7710
81	0.2010 8797	0.1649 1925	0.1353 2153	0.1110 8917	0.0912 3990
82	0.1971 4507	0.1612 9022	0.1320 2101	0.1081 1598	0.0885 8243
83	0.1932 7948	0.1577 4105	0.1288 0098	0.1052 2237	0.0860 0236
84	0.1894 8968	0.1542 6997	0.1256 5949	0.1024 0620	0.0834 9743
85	0.1857 7420	0.1508 7528	0.1225 9463	0.0996 6540	0.0810 6547
86	0.1821 3157	0.1475 5528	0.1196 0452	0.0969 9795	0.0787 0434
87	0.1785 6036	0.1443 0835	0.1166 8733	0.0944 0190	0.0764 1198
88	0.1750 5918	0.1411 3286	0.1138 4130	0.0918 7533	0.0741 8639
89	0.1716 2665	0.1380 2724	0.1110 6468	0.0894 1638	0.0720 2562
90	0.1682 6142	0.1349 8997	0.1083 5579	0.0870 2324	0.0699 2779
91	0.1649 6217	0.1320 1953	0.1057 1296	0.0846 9415	0.0678 9105
92	0.1617 2762	0.1291 1445	0.1031 3460	0.0824 2740	0.0659 1364
93	0.1585 5649	0.1262 7331	0.1006 1912	0.0802 2131	0.0639 9383
94	0.1554 4754	0.1234 9468	0.0981 6500	0.0780 7427	0.0621 2993
95	0.1523 9955	0.1207 7719	0.0957 7073	0.0759 8469	0.0603 2032
96	0.1494 1132	0.1181 1950	0.0934 3486	0.0739 5104	0.0585 6342
97	0.1464 8169	0.1155 2029	0.0911 5596	0.0719 7181	0.0568 5769
98	0.1436 0950	0.1129 7828	0.0889 3264	0.0700 4556	0.0552 0164
99	0.1407 9363	0.1104 9221	0.0867 6355	0.0681 7086	0.0535 9383
100	0.1380 3297	0.1080 6084	0.0846 4537	0.0663 4634	0.0520 3284

TABLE IV. The Present Value of 1 at Compound Interest

$$v^n = (1+i)^{-n}$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$
1	0.9661 8357	0.9615 3846	0.9569 3780	0.9523 8095	0.9478 6730
2	0.9335 1070	0.9245 5621	0.9157 2995	0.9070 2948	0.8984 5242
3	0.9019 4271	0.8889 9636	0.8762 9660	0.8638 3760	0.8516 1366
4	0.8714 4223	0.8548 0419	0.8385 6134	0.8227 0247	0.8072 1674
5	0.8419 7317	0.8219 2711	0.8024 5105	0.7835 2617	0.7651 3435
6	0.8135 0064	0.7903 1453	0.7678 9574	0.7462 1540	0.7252 4583
7	0.7859 9096	0.7599 1781	0.7348 2846	0.7106 8133	0.6874 3681
8	0.7594 1156	0.7306 9021	0.7031 8513	0.6768 3936	0.6515 9887
9	0.7337 3097	0.7025 8674	0.6729 0443	0.6446 0892	0.6176 2926
10	0.7089 1881	0.6755 6417	0.6439 2768	0.6139 1325	0.5854 3058
11	0.6849 4571	0.6495 8093	0.6161 9874	0.5846 7929	0.5549 1050
12	0.6617 8330	0.6245 9705	0.5896 6386	0.5568 3742	0.5259 8152
13	0.6394 0415	0.6005 7409	0.5642 7164	0.5303 2135	0.4985 6068
14	0.6177 8179	0.5774 7508	0.5399 7286	0.5050 6795	0.4725 6937
15	0.5968 9062	0.5552 6450	0.5167 2044	0.4810 1710	0.4479 3305
16	0.5767 0591	0.5339 0818	0.4944 6932	0.4581 1152	0.4245 8109
17	0.5572 0378	0.5133 7325	0.4731 7639	0.4362 9669	0.4024 4653
18	0.5383 6114	0.4936 2812	0.4528 0037	0.4155 2065	0.3814 6590
19	0.5201 5569	0.4746 4242	0.4333 0179	0.3957 3396	0.3615 7906
20	0.5025 6588	0.4563 8695	0.4146 4286	0.3768 8948	0.3427 2896
21	0.4855 7090	0.4388 3360	0.3967 8743	0.3589 4236	0.3248 6158
22	0.4691 5063	0.4219 5539	0.3797 0089	0.3418 4987	0.3079 2567
23	0.4532 8563	0.4057 2633	0.3633 5013	0.3255 7131	0.2918 7267
24	0.4379 5713	0.3901 2147	0.3477 0347	0.3100 6791	0.2766 5656
25	0.4231 4699	0.3751 1680	0.3327 3060	0.2953 0277	0.2622 3370
26	0.4088 3767	0.3606 8923	0.3184 0248	0.2812 4073	0.2485 6275
27	0.3950 1224	0.3468 1657	0.3046 9137	0.2678 4832	0.2356 0450
28	0.3816 5434	0.3334 7747	0.2915 7069	0.2550 9364	0.2233 2181
29	0.3687 4815	0.3206 5141	0.2790 1502	0.2429 4632	0.2116 7944
30	0.3562 7841	0.3083 1867	0.2670 0002	0.2313 7745	0.2006 4402
31	0.3442 3035	0.2964 6026	0.2555 0241	0.2203 5947	0.1901 8390
32	0.3325 8971	0.2850 5794	0.2444 9991	0.2098 6617	0.1802 6910
33	0.3213 4271	0.2740 9417	0.2339 7121	0.1998 7254	0.1708 7119
34	0.3104 7605	0.2635 5209	0.2238 9589	0.1903 5480	0.1619 6321
35	0.2999 7686	0.2534 1547	0.2142 5444	0.1812 9029	0.1535 1963
36	0.2898 3272	0.2436 6872	0.2050 2817	0.1726 5741	0.1455 1624
37	0.2800 3161	0.2342 9685	0.1961 9921	0.1644 3563	0.1379 3008
38	0.2705 6194	0.2252 8543	0.1877 5044	0.1566 0536	0.1307 3941
39	0.2614 1250	0.2166 2061	0.1796 6549	0.1491 4797	0.1239 2362
40	0.2525 7247	0.2082 8904	0.1719 2870	0.1420 4568	0.1174 6314
41	0.2440 3137	0.2002 7793	0.1645 2507	0.1352 8160	0.1113 3947
42	0.2357 7910	0.1925 7493	0.1574 4026	0.1288 3962	0.1055 3504
43	0.2278 0590	0.1851 6820	0.1506 6054	0.1227 0440	0.1000 3322
44	0.2201 0231	0.1780 4635	0.1441 7276	0.1168 6133	0.0948 1822
45	0.2126 5924	0.1711 9841	0.1379 6437	0.1112 9651	0.0898 7509
46	0.2054 6787	0.1646 1386	0.1320 2332	0.1059 9668	0.0851 8965
47	0.1985 1968	0.1582 8256	0.1263 3810	0.1009 4921	0.0807 4849
48	0.1918 0645	0.1521 9476	0.1208 9771	0.0961 4211	0.0765 3885
49	0.1853 2024	0.1463 4112	0.1156 9158	0.0915 6391	0.0725 4867
50	0.1790 5337	0.1407 1262	0.1107 0965	0.0872 0373	0.0687 6652

TABLE IV. The Present Value of 1 at Compound Interest

$$v^n = (1+i)^{-n}$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$
51	0.1729 9843	0.1353 0059	0.1059 4225	0.0830 5117	0.0651 8153
52	0.1671 4824	0.1300 9672	0.1013 8014	0.0790 9635	0.0617 8344
53	0.1614 9589	0.1250 9300	0.0970 1449	0.0753 2986	0.0585 6250
54	0.1560 3467	0.1202 8173	0.0928 3683	0.0717 4272	0.0555 0948
55	0.1507 5814	0.1156 5551	0.0888 3907	0.0683 2640	0.0526 1562
56	0.1456 6004	0.1112 0722	0.0850 1347	0.0650 7276	0.0498 7263
57	0.1407 3433	0.1069 3002	0.0813 5260	0.0619 7406	0.0472 7263
58	0.1359 7520	0.1028 1733	0.0778 4938	0.0590 2291	0.0448 0818
59	0.1313 7701	0.0988 6282	0.0744 9701	0.0562 1230	0.0424 7221
60	0.1269 3431	0.0950 6040	0.0712 8901	0.0535 3552	0.0402 5802
61	0.1226 4184	0.0914 0423	0.0682 1915	0.0509 8621	0.0381 5926
62	0.1184 9453	0.0878 8868	0.0652 8148	0.0485 5830	0.0361 6992
63	0.1144 8747	0.0845 0835	0.0624 7032	0.0462 4600	0.0342 8428
64	0.1106 1591	0.0812 5803	0.0597 8021	0.0440 4381	0.0324 9695
65	0.1068 7528	0.0781 3272	0.0572 0594	0.0419 4648	0.0308 0279
66	0.1032 6114	0.0751 2762	0.0547 4253	0.0399 4903	0.0291 9696
67	0.0997 6922	0.0722 3809	0.0523 8519	0.0380 4670	0.0276 7485
68	0.0963 9538	0.0694 5970	0.0501 2937	0.0362 3495	0.0262 3208
69	0.0931 3563	0.0667 8818	0.0479 7069	0.0345 0948	0.0248 6453
70	0.0899 8612	0.0642 1940	0.0459 0497	0.0328 6617	0.0235 6828
71	0.0869 4311	0.0617 4942	0.0439 2820	0.0313 0111	0.0223 3960
72	0.0840 0300	0.0593 7445	0.0420 3655	0.0298 1058	0.0211 7498
73	0.0811 6232	0.0570 9081	0.0402 2637	0.0283 9103	0.0200 7107
74	0.0784 1770	0.0548 9501	0.0384 9413	0.0270 3908	0.0190 2471
75	0.0757 6590	0.0527 8367	0.0368 3649	0.0257 5150	0.0180 3290
76	0.0732 0376	0.0507 5353	0.0352 5023	0.0245 2524	0.0170 9279
77	0.0707 2827	0.0488 0147	0.0337 3228	0.0233 5737	0.0162 0170
78	0.0683 3650	0.0469 2449	0.0322 7969	0.0222 4512	0.0153 5706
79	0.0660 2560	0.0451 1970	0.0308 8965	0.0211 8582	0.0145 5646
80	0.0637 9285	0.0433 8433	0.0295 5948	0.0201 7698	0.0137 9759
81	0.0616 3561	0.0417 1570	0.0282 8658	0.0192 1617	0.0130 7828
82	0.0595 5131	0.0401 1125	0.0270 6850	0.0183 0111	0.0123 9648
83	0.0575 3750	0.0385 6851	0.0259 0287	0.0174 2963	0.0117 5022
84	0.0555 9178	0.0370 8510	0.0247 8744	0.0165 9965	0.0111 3765
85	0.0537 1187	0.0356 5875	0.0237 2003	0.0158 0919	0.0105 5701
86	0.0518 9553	0.0342 8726	0.0226 9860	0.0150 5637	0.0100 0664
87	0.0501 4060	0.0329 6852	0.0217 2115	0.0143 3940	0.0094 8497
88	0.0484 4503	0.0317 0050	0.0207 8579	0.0136 5657	0.0089 9049
89	0.0468 0679	0.0304 8125	0.0198 9070	0.0130 0626	0.0085 2180
90	0.0452 2395	0.0293 0890	0.0190 3417	0.0123 8691	0.0080 7753
91	0.0436 9464	0.0281 8163	0.0182 1451	0.0117 9706	0.0076 5643
92	0.0422 1704	0.0270 9772	0.0174 3016	0.0112 3530	0.0072 5728
93	0.0407 8941	0.0260 5550	0.0166 7958	0.0107 0028	0.0068 7894
94	0.0394 1006	0.0250 5337	0.0159 6132	0.0101 9074	0.0065 2032
95	0.0380 7735	0.0240 8978	0.0152 7399	0.0097 0547	0.0061 8040
96	0.0367 8971	0.0231 6325	0.0146 1626	0.0092 4331	0.0058 5820
97	0.0355 4562	0.0222 7235	0.0139 8685	0.0088 0315	0.0055 5279
98	0.0343 4359	0.0214 1572	0.0133 8454	0.0083 8395	0.0052 6331
99	0.0331 8221	0.0205 9204	0.0128 0817	0.0079 8471	0.0049 8892
100	0.0320 6011	0.0198 0004	0.0122 5663	0.0076 0449	0.0047 2883

TABLE IV. The Present Value of 1 at Compound Interest

$$v^n = (1+i)^{-n}$$

n	6%	$6\frac{1}{2}$%	7%	$7\frac{1}{2}$%	8%
1	0.9433 9623	0.9389 6714	0.9345 7944	0.9302 3256	0.9259 2593
2	0.8899 9644	0.8816 5928	0.8734 3872	0.8653 3261	0.8573 3882
3	0.8396 1928	0.8278 4909	0.8162 9788	0.8049 6057	0.7938 3224
4	0.7920 9366	0.7773 2309	0.7628 9521	0.7488 0053	0.7350 2985
5	0.7472 5817	0.7298 8084	0.7129 8618	0.6965 5863	0.6805 8320
6	0.7049 6054	0.6853 3412	0.6663 4222	0.6479 6152	0.6301 6963
7	0.6650 5711	0.6435 0621	0.6227 4974	0.6027 5490	0.5834 9040
8	0.6274 1237	0.6042 3119	0.5820 0910	0.5607 0223	0.5402 6888
9	0.5918 9846	0.5673 5323	0.5439 3374	0.5215 8347	0.5002 4897
10	0.5583 9478	0.5327 2604	0.5083 4929	0.4851 9393	0.4631 9349
11	0.5267 8753	0.5002 1224	0.4750 9280	0.4513 4319	0.4288 8286
12	0.4969 6936	0.4696 8285	0.4440 1196	0.4198 5413	0.3971 1376
13	0.4688 3902	0.4410 1676	0.4149 6445	0.3905 6198	0.3676 9792
14	0.4423 0096	0.4141 0025	0.3878 1724	0.3633 1347	0.3404 6104
15	0.4172 6506	0.3888 2652	0.3624 4602	0.3379 6602	0.3152 4170
16	0.3936 4628	0.3650 9533	0.3387 3460	0.3143 8699	0.2918 9047
17	0.3713 6442	0.3428 1251	0.3165 7439	0.2924 5302	0.2702 6895
18	0.3503 4379	0.3218 8969	0.2958 6392	0.2720 4932	0.2502 4903
19	0.3305 1301	0.3022 4384	0.2765 0832	0.2530 6913	0.2317 1206
20	0.3118 0473	0.2837 9703	0.2584 1900	0.2354 1315	0.2145 4821
21	0.2941 5540	0.2664 7608	0.2415 1309	0.2189 8897	0.1986 5575
22	0.2775 0510	0.2502 1228	0.2257 1317	0.2037 1067	0.1839 4051
23	0.2617 9726	0.2349 4111	0.2109 4688	0.1894 9830	0.1703 1528
24	0.2469 7855	0.2206 0198	0.1971 4662	0.1762 7749	0.1576 9934
25	0.2329 9863	0.2071 3801	0.1842 4918	0.1639 7906	0.1460 1790
26	0.2198 1003	0.1944 9579	0.1721 9549	0.1525 3866	0.1352 0176
27	0.2073 6795	0.1826 2515	0.1609 3037	0.1418 9643	0.1251 8682
28	0.1956 3014	0.1714 7902	0.1504 0221	0.1319 9668	0.1159 1372
29	0.1845 5674	0.1610 1316	0.1405 6282	0.1227 8761	0.1073 2752
30	0.1741 1013	0.1511 8607	0.1313 6712	0.1142 2103	0.0993 7733
31	0.1642 5484	0.1419 5875	0.1227 7301	0.1062 5212	0.0920 1605
32	0.1549 5740	0.1332 9460	0.1147 4113	0.0988 3918	0.0852 0005
33	0.1461 8622	0.1251 5925	0.1072 3470	0.0919 4343	0.0788 8893
34	0.1379 1153	0.1175 2042	0.1002 1934	0.0855 2877	0.0730 4531
35	0.1301 0522	0.1103 4781	0.0936 6294	0.0795 6164	0.0676 3454
36	0.1227 4077	0.1036 1297	0.0875 3546	0.0740 1083	0.0626 2458
37	0.1157 9318	0.0972 8917	0.0818 0884	0.0688 4729	0.0579 8572
38	0.1092 3885	0.0913 5134	0.0764 5686	0.0640 4399	0.0536 9048
39	0.1030 5552	0.0857 7590	0.0714 5501	0.0595 7580	0.0497 1341
40	0.0972 2219	0.0805 4075	0.0667 8038	0.0554 1935	0.0460 3093
41	0.0917 1905	0.0756 2512	0.0624 1157	0.0515 5288	0.0426 2123
42	0.0865 2740	0.0710 0950	0.0583 2857	0.0479 5617	0.0394 6411
43	0.0816 2962	0.0666 7559	0.0545 1268	0.0446 1039	0.0365 4084
44	0.0770 0908	0.0626 0619	0.0509 4643	0.0414 9804	0.0338 3411
45	0.0726 5007	0.0587 8515	0.0476 1349	0.0386 0283	0.0313 2788
46	0.0685 3781	0.0551 9733	0.0444 9859	0.0359 0961	0.0290 0730
47	0.0646 5831	0.0518 2848	0.0415 8747	0.0334 0428	0.0268 5861
48	0.0609 9840	0.0486 6524	0.0388 6679	0.0310 7375	0.0248 6908
49	0.0575 4566	0.0456 9506	0.0363 2410	0.0289 0582	0.0230 2693
50	0.0542 8836	0.0429 0616	0.0339 4776	0.0268 8913	0.0213 2123

TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{8}\%$	1%
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0033 3333	2.0041 6667	2.0050 0000	2.0087 5000	2.0100 0000
3	3.0100 1111	3.0125 1736	3.0150 2500	3.0263 2656	3.0301 0000
4	4.0200 4448	4.0250 6952	4.0301 0013	4.0528 0692	4.0604 0100
5	5.0334 4463	5.0418 4064	5.0502 5063	5.0882 6898	5.1010 0501
6	6.0502 2278	6.0628 4831	6.0755 0188	6.1327 9133	6.1520 1506
7	7.0703 9019	7.0881 1018	7.1058 7939	7.1864 5326	7.2135 3521
8	8.0939 5816	8.1176 4397	8.1414 0879	8.2493 3472	8.2856 7056
9	9.1209 3802	9.1514 6749	9.1821 1583	9.3215 1640	9.3685 2727
10	10.1513 4114	10.1895 9860	10.2280 2641	10.4030 7967	10.4622 1254
11	11.1851 7895	11.2320 5526	11.2791 6654	11.4941 0662	11.5668 3467
12	12.2224 6288	12.2788 5549	12.3355 6237	12.5946 8005	12.6825 0301
13	13.2632 0442	13.3300 1739	13.3972 4018	13.7048 8350	13.8093 2804
14	14.3074 1510	14.3855 5913	14.4642 2639	14.8248 0123	14.9474 2132
15	15.3551 0648	15.4454 9896	15.5365 4752	15.9545 1824	16.0968 9554
16	16.4062 9017	16.5098 5520	16.6142 3026	17.0941 2028	17.2578 6449
17	17.4609 7781	17.5786 4627	17.6973 0141	18.2436 9383	18.4304 4314
18	18.5191 8107	18.6518 9063	18.7857 8791	19.4033 2615	19.6147 4757
19	19.5809 1167	19.7296 0684	19.8797 1685	20.5731 0526	20.8108 9504
20	20.6461 8137	20.8118 1353	20.9791 1544	21.7531 1993	22.0190 0399
21	21.7150 0198	21.8985 2942	22.0840 1101	22.9434 5973	23.2391 9403
22	22.7873 8532	22.9897 7330	23.1944 3107	24.1442 1500	24.4715 8598
23	23.8633 4327	24.0855 6402	24.3104 0322	25.3554 7688	25.7163 0183
24	24.9428 8775	25.1859 2054	25.4319 5524	26.5773 3730	26.9734 6485
25	26.0260 3071	26.2908 6187	26.5591 1502	27.8098 8900	28.2431 9950
26	27.1127 8414	27.4004 0713	27.6919 1059	29.0532 2553	29.5256 3150
27	28.2031 6009	28.5145 7549	28.8303 7015	30.3074 4126	30.8208 8781
28	29.2971 7062	29.6333 8622	29.9745 2200	31.5726 3137	32.1290 9669
29	30.3948 2786	30.7568 5867	31.1243 9461	32.8488 9189	33.4503 8766
30	31.4961 4395	31.8850 1224	32.2800 1658	34.1363 1970	34.7848 9153
31	32.6011 3110	33.0178 6646	33.4414 1666	35.4350 1249	36.1327 4045
32	33.7098 0154	34.1554 4090	34.6086 2375	36.7450 6885	37.4940 6785
33	34.8221 6754	35.2977 5524	35.7816 6686	38.0665 8820	38.8690 0853
34	35.9382 4143	36.4448 2922	36.9605 7520	39.3996 7085	40.2576 9862
35	37.0580 3557	37.5966 8268	38.1453 7807	40.7444 1797	41.6602 7560
36	38.1815 6236	38.7533 3552	39.3361 0496	42.1009 3163	43.0768 7836
37	39.3088 3423	39.9148 0775	40.5327 8549	43.4693 1478	44.5076 4714
38	40.4398 6368	41.0811 1945	41.7354 4942	44.8496 7128	45.9527 2361
39	41.5746 6322	42.2522 9078	42.9441 2666	46.2421 0591	47.4122 5085
40	42.7132 4543	43.4283 4199	44.1588 4730	47.6467 2433	48.8863 7336
41	43.8556 2292	44.6092 9342	45.3796 4153	49.0636 3317	50.3752 3709
42	45.0018 0833	45.7951 6548	46.6065 3974	50.4929 3996	51.8789 8946
43	46.1518 1436	46.9859 7866	47.8395 7244	51.9347 5319	53.3977 7936
44	47.3056 5374	48.1817 5358	49.0787 7030	53.3891 8228	54.9317 5715
45	48.4633 3925	49.3825 1088	50.3241 6415	54.8563 3762	56.4810 7472
46	49.6248 8371	50.5882 7134	51.5757 8497	56.3363 3058	58.0458 8547
47	50.7902 9999	51.7990 5581	52.8336 6390	57.8292 7347	59.6263 4432
48	51.9596 0099	53.0148 8521	54.0978 3222	59.3352 7961	61.2226 0777
49	53.1327 9966	54.2357 8056	55.3683 2138	60.8544 6331	62.8348 3385
50	54.3099 0899	55.4617 6298	56.6451 6299	62.3869 3986	64.4631 8218

TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

n	$\frac{1}{8}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{8}\%$	1
51	55.4909 4202	56.6928 5366	57.9283 8880	63.9328 2559	66.1078 1401
52	56.6759 1183	57.9290 7388	59.2180 3075	65.4922 3781	67.7688 9215
53	57.8648 3154	59.1704 4503	60.5141 2090	67.0652 9489	69.4465 8107
54	59.0577 1431	60.4169 8855	61.8166 9150	68.6521 1622	71.1410 4688
55	60.2545 7336	61.6687 2600	63.1257 7496	70.2528 2224	72.8524 5735
56	61.4554 2194	62.9256 7902	64.4414 0384	71.8675 3443	74.5809 8192
57	62.6602 7334	64.1878 6935	65.7636 1086	73.4963 7536	76.3267 9174
58	63.8691 4092	65.4553 1881	67.0924 2891	75.1394 6864	78.0900 5966
59	65.0820 3806	66.7280 4930	68.4278 9105	76.7969 3900	79.8709 6025
60	66.2989 7818	68.0060 8284	69.7700 3051	78.4689 1221	81.6696 6986
61	67.5199 7478	69.2894 4152	71.1188 8066	80.1555 1519	83.4863 6655
62	68.7450 4136	70.5781 4753	72.4744 7507	81.8568 7595	85.3212 3022
63	69.9741 9150	71.8722 2314	73.8368 4744	83.5731 2362	87.1744 4252
64	71.2074 3880	73.1716 9074	75.2060 3168	85.3043 8845	89.0461 8695
65	72.4447 9693	74.4765 7278	76.5820 6184	87.0508 0185	90.9366 4882
66	73.6862 7959	75.7868 9184	77.9649 7215	88.8124 9636	92.8460 1531
67	74.9319 0052	77.1026 7055	79.3547 9701	90.5896 0571	94.7744 7546
68	76.1816 7352	78.4239 3168	80.7515 7099	92.3822 6476	96.7222 2021
69	77.4356 1243	79.7506 9806	82.1553 2885	94.1906 0957	98.6894 4242
70	78.6937 3114	81.0829 9264	83.5661 0549	96.0147 7741	100.6763 3684
71	79.9560 4358	82.4208 3844	84.9839 3602	97.8549 0671	102.6831 0021
72	81.2225 6372	83.7642 5860	86.4088 5570	99.7111 3714	104.7099 3121
73	82.4933 0560	85.1132 7634	87.8408 9998	101.5836 0959	106.7570 3052
74	83.7682 8329	86.4679 1500	89.2801 0448	103.4724 6618	108.8246 0083
75	85.0475 1090	87.8281 9797	90.7265 0500	105.3778 5025	110.9128 4684
76	86.3310 0260	89.1941 4880	92.1801 3752	107.2999 0644	113.0219 7530
77	87.6187 7261	90.5657 9109	93.6410 3821	109.2387 8063	115.1521 9506
78	88.9108 3519	91.9431 4855	95.1092 4340	111.1946 1996	117.3037 1701
79	90.2072 0464	93.3262 4500	96.5847 8962	113.1675 7288	119.4767 5418
80	91.5078 9532	94.7151 0436	98.0677 1357	115.1577 8914	121.6715 2172
81	92.8129 2164	96.1097 5062	99.5580 5214	117.1654 1980	123.8882 3694
82	94.1222 9804	97.5102 0792	101.0558 4240	119.1906 1722	126.1271 1931
83	95.4360 3904	98.9165 0045	102.5611 2161	121.2335 3512	128.3883 9050
84	96.7541 5917	100.3286 5254	104.0739 2722	123.2943 2855	130.6722 7440
85	98.0766 7303	101.7466 8859	105.5942 9685	125.3731 5393	132.9789 9715
86	99.4035 9527	103.1706 3312	107.1222 6834	127.4701 6903	135.3087 8712
87	100.7349 4059	104.6005 1076	108.6578 7968	129.5855 3301	137.6618 7499
88	102.0707 2373	106.0363 4622	110.2011 6908	131.7194 0642	140.0384 9374
89	103.4109 5947	107.4781 6433	111.7521 7492	133.8719 5123	142.4388 7868
90	104.7556 6267	108.9259 9002	113.3109 3580	136.0433 3080	144.8632 6746
91	106.1048 4821	110.3798 4831	114.8774 9048	138.2337 0994	147.3119 0014
92	107.4585 3104	111.8397 6434	116.4518 7793	140.4432 5491	149.7850 1914
93	108.8167 2614	113.3057 6336	118.0341 3732	142.6721 3339	152.2828 6933
94	110.1794 4856	114.7778 7071	119.6243 0800	144.9205 1455	154.8056 9803
95	111.5467 1339	116.2561 1184	121.2224 2954	147.1885 6906	157.3537 5501
96	112.9185 3577	117.7405 1230	122.8285 4169	149.4764 6903	159.9272 9256
97	114.2949 3089	119.2310 9777	124.4426 8440	151.7843 8813	162.5265 6548
98	115.6759 1399	120.7278 9401	126.0648 9782	154.1125 0153	165.1518 3114
99	117.0615 0037	122.2309 2690	127.6952 2231	156.4609 8592	167.8033 4945
100	118.4517 0537	123.7402 2243	129.3336 9842	158.8300 1955	170.4813 8294

TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0112 5000	2.0125 0000	2.0137 5000	2.0150 0000	2.0175 0000
3	3.0338 7656	3.0376 5625	3.0414 3906	3.0452 2500	3.0528 0625
4	4.0680 0767	4.0756 2695	4.0832 5885	4.0909 0338	4.1062 3036
5	5.1137 7276	5.1265 7229	5.1394 0366	5.1522 6693	5.1780 8938
6	6.1713 0270	6.1906 5444	6.2100 7046	6.2295 5093	6.2687 0596
7	7.2407 2986	7.2680 3762	7.2954 5893	7.3229 9419	7.3784 0831
8	8.3221 8807	8.3588 8809	8.3957 7149	8.4328 3911	8.5075 3045
9	9.4158 1269	9.4633 7420	9.5112 1335	9.5593 3169	9.6564 1224
10	10.5217 4058	10.5816 6637	10.6419 9253	10.7027 2167	10.8253 9945
11	11.6401 1016	11.7139 3720	11.7883 1993	11.8632 6249	12.0148 4394
12	12.7710 6140	12.8603 6142	12.9504 0933	13.0412 1143	13.2251 0371
13	13.9147 3584	14.0211 1594	14.1284 7745	14.2368 2960	14.4565 4303
14	15.0712 7662	15.1963 7988	15.3227 4402	15.4503 8205	15.7095 3253
15	16.2408 2848	16.3863 3463	16.5334 3175	16.6821 3778	16.9844 4935
16	17.4235 3780	17.5911 6382	17.7607 6644	17.9323 6984	18.2816 7721
17	18.6195 5260	18.8110 5336	19.0049 7697	19.2013 5539	19.6016 0656
18	19.8290 2257	20.0461 9153	20.2662 9541	20.4893 7572	20.9446 3468
19	21.0520 9907	21.2967 6893	21.5449 5697	21.7967 1636	22.3111 6578
20	22.2889 3519	22.5629 7854	22.8412 0013	23.1236 6710	23.7016 1119
21	23.5396 8571	23.8450 1577	24.1552 6663	24.4705 2211	25.1163 8938
22	24.8045 0717	25.1430 7847	25.4874 0155	25.8375 7994	26.5559 2620
23	26.0835 5788	26.4573 6695	26.8378 5332	27.2251 4364	28.0206 5490
24	27.3769 9790	27.7880 8403	28.2068 7380	28.6335 2080	29.5110 1637
25	28.6849 8913	29.1354 3508	29.5947 1832	30.0630 2361	31.0274 5915
26	30.0076 9526	30.4996 2802	31.0016 4569	31.5139 6896	32.5704 3969
27	31.3452 8183	31.8808 7337	32.4279 1832	32.9866 7850	34.1404 2238
28	32.6979 1625	33.2793 8429	33.8738 0220	34.4814 7867	35.7378 7977
29	34.0657 6781	34.6953 7659	35.3395 6698	35.9987 0085	37.3632 9267
30	35.4490 0769	36.1290 6880	36.8254 8602	37.5386 8137	39.0171 5029
31	36.8478 0903	37.5806 8216	38.3318 3646	39.1017 6159	40.6999 5042
32	38.2623 4688	39.0504 4069	39.8588 9921	40.6882 8801	42.4121 9955
33	39.6927 9829	40.5385 7120	41.4069 5907	42.2986 1233	44.1544 1305
34	41.1393 4227	42.0453 0334	42.9763 0476	43.9330 9152	45.9271 1527
35	42.6021 5987	43.5708 6963	44.5672 2895	45.5920 8789	47.7308 3979
36	44.0814 3417	45.1155 0550	46.1800 2835	47.2759 6921	49.5661 2949
37	45.5773 5030	46.6794 4932	47.8150 0374	48.9851 0874	51.4335 3675
38	47.0900 9549	48.2926 4243	49.4724 6004	50.7198 8538	53.3336 2365
39	48.6198 5906	49.8862 2921	51.1527 0636	52.4806 8366	55.2669 6206
40	50.1668 3248	51.4895 5708	52.8560 5608	54.2678 9391	57.2341 3390
41	51.7312 0934	53.1331 7654	54.5828 2685	56.0819 1232	59.2357 3124
42	53.3131 8545	54.7973 4125	56.3333 4072	57.9231 4100	61.2723 5654
43	54.9129 5879	56.4823 0801	58.1079 2415	59.7919 8812	63.3446 2278
44	56.5307 2957	58.1883 3687	59.9069 0811	61.6888 6794	65.4531 5367
45	58.1667 0028	59.9156 9108	61.7306 2810	63.6142 0096	67.5985 8386
46	59.8210 7566	61.6646 3721	63.5794 2423	65.5684 1398	69.7815 5908
47	61.4940 6276	63.4354 4518	65.4536 4131	67.5519 4018	72.0027 3637
48	63.1858 7097	65.2283 8824	67.3536 2888	69.5652 1929	74.2627 8425
49	64.8967 1201	67.0437 4310	69.2797 4128	71.6086 9758	76.5623 8298
50	66.6268 0002	68.8817 8989	71.2323 3772	73.6828 2804	78.9022 2468

TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
51	68.3763 5152	70.7428 1226	73.2117 8237	75.7880 7046	81.2830 1361
52	70.1455 8548	72.6270 0741	75.2184 4437	77.9248 9152	83.7054 6635
53	71.9347 2332	74.5349 3613	77.2526 9798	80.0937 6489	86.1703 1201
54	73.7439 8895	76.4666 2283	79.3149 2258	82.2951 7136	88.6782 9247
55	75.5736 0883	78.4224 5562	81.4055 0277	84.5295 9893	91.2301 6259
56	77.4238 1193	80.4027 3631	83.5248 2843	86.7975 4292	93.8266 9043
57	79.2948 2981	82.4077 7052	85.6732 9482	89.0995 0606	96.4686 5752
58	81.1868 9665	84.4378 6765	87.8513 0262	91.4359 9865	99.1568 5902
59	83.1002 4923	86.4933 4099	90.0592 5804	93.8075 3863	101.8921 0405
60	85.0351 2704	88.5745 0776	92.2975 7283	96.2146 5171	104.6752 1588
61	86.9917 7222	90.6816 8910	94.5666 6446	98.6578 7149	107.5070 3215
62	88.9704 2966	92.8152 1022	96.8669 5610	101.1377 3956	110.3884 0522
63	90.9713 4699	94.9754 0034	99.1988 7674	103.6548 0565	113.3202 0231
64	91.9947 7464	97.1625 9285	101.5628 6130	106.2096 2774	116.3033 0585
65	95.0409 6586	99.3771 2526	103.9593 5064	108.8027 7215	119.3386 1370
66	97.1101 7672	101.6193 3933	106.3887 9171	111.4348 1374	122.4270 3941
67	99.2026 6621	103.8895 8107	108.8516 3760	114.1063 3594	125.5695 1263
68	101.3186 9621	106.1882 0083	111.3483 4761	116.8179 3098	128.7669 7910
69	103.4585 3154	108.5155 5334	113.8793 8739	119.5701 9995	132.0204 0124
70	105.6224 4002	110.8719 9776	116.4452 2897	122.3637 5295	135.3307 5826
71	107.8106 9247	113.2578 9773	119.0463 5087	125.1992 0924	138.6990 4653
72	110.0235 6276	115.6736 2145	121.6832 3819	128.0771 9738	142.1262 7984
73	112.2613 2784	118.1195 4172	124.3563 8272	130.9983 5534	145.6134 8974
74	114.5242 6778	120.5960 3599	127.0662 8298	133.9633 3067	149.1617 2581
75	116.8126 6579	123.1034 8644	129.8134 4437	136.9727 8063	152.7720 5601
76	119.1268 0828	125.6422 8002	132.5983 7923	140.0273 7234	156.4455 6699
77	121.4669 8487	128.2128 0852	135.4216 0695	143.1277 8292	160.1833 6441
78	123.8334 8845	130.8151 6863	138.2836 5404	146.2746 9967	163.9865 7329
79	126.2266 1520	133.4506 6199	141.1850 5429	149.4688 2016	167.8563 3832
80	128.6466 6462	136.1187 9526	144.1263 4878	152.7108 5247	171.7938 2424
81	131.0939 3960	138.8202 8020	147.1080 8608	156.0015 1525	175.8002 1617
82	133.5687 4642	141.5555 3370	150.1308 2226	159.3415 3798	179.8767 1995
83	136.0713 9481	144.3249 7787	153.1951 2107	162.7316 6105	184.0245 6255
84	138.6021 9801	147.1290 4010	156.3015 5398	166.1726 3597	188.2449 9239
85	141.1614 7273	149.9681 5310	159.4507 0035	169.6652 2551	192.5392 7976
86	143.7495 3930	152.8427 5501	162.6431 4748	173.2102 0389	196.9087 1716
87	146.3667 2162	155.7532 8945	165.8794 9076	176.8083 5695	201.3546 1971
88	149.0133 4724	158.7002 0557	169.1603 3375	180.4604 8230	205.8783 2555
89	151.6897 4739	161.6839 5814	172.4862 8834	184.1673 8954	210.4811 9625
90	154.3962 5705	164.7050 0762	175.8579 7481	187.9299 0038	215.1646 1718
91	157.1332 1494	167.7638 2021	179.2760 2196	191.7488 4889	219.9299 9798
92	159.9009 6361	170.8608 6796	182.7410 6726	195.6250 8162	224.7787 7295
93	162.6998 4945	173.9966 2881	186.2537 5694	199.5594 5784	229.7124 0148
94	165.5302 2276	177.1715 8667	189.8147 4610	203.5528 4971	234.7323 6850
95	168.3924 3776	180.3862 3151	193.4246 9886	207.6061 4246	239.8401 8495
96	171.2868 5269	183.6410 5940	197.0842 8847	211.7202 3459	245.0373 8819
97	174.2138 2978	186.9365 7264	200.7941 9743	215.8960 3811	250.3255 4248
98	177.1737 3537	190.2732 7980	204.5551 1765	220.1344 7868	255.7062 3947
99	180.1669 3989	193.6516 9580	208.3677 5051	224.4364 9586	261.1810 9866
100	183.1938 1796	197.0723 4200	212.2328 0708	228.8030 4330	266.7517 6789

TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

n	2%	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0200 0000	2.0225 0000	2.0250 0000	2.0275 0000	2.0300 0000
3	3.0604 0000	3.0680 0625	3.0756 2500	3.0832 5625	3.0909 0000
4	4.1216 0800	4.1370 3639	4.1525 1563	4.1680 4580	4.1836 2700
5	5.2040 4016	5.2301 1971	5.2563 2852	5.2826 6706	5.3091 3581
6	6.3081 2096	6.3477 9740	6.3877 3673	6.4279 4040	6.4684 0988
7	7.4342 8338	7.4906 2284	7.5474 3015	7.6047 0876	7.6624 6218
8	8.5829 6905	8.6591 6186	8.7361 1590	8.8138 3825	8.8923 3605
9	9.7546 2843	9.8539 9300	9.9545 1880	10.0562 1880	10.1591 0613
10	10.9497 2100	11.0757 0784	11.2033 8177	11.3327 6482	11.4638 7931
11	12.1687 1542	12.3249 1127	12.4834 6631	12.6444 1585	12.8077 9569
12	13.4120 8973	13.6022 2177	13.7955 5297	13.9921 3729	14.1920 2956
13	14.6803 3152	14.9082 7176	15.1404 4179	15.3769 2107	15.6177 9045
14	15.9739 3815	16.2437 0788	16.5189 5284	16.7997 8639	17.0863 2416
15	17.2934 1692	17.6091 9130	17.9319 2666	18.2617 8052	18.5989 1389
16	18.6392 8525	19.0053 9811	19.3802 2483	19.7639 7948	20.1568 8130
17	20.0120 7096	20.4330 1957	20.8647 3045	21.3074 8892	21.7615 8774
18	21.4123 1238	21.8927 6251	22.3863 4871	22.8934 4487	23.4144 3537
19	22.8405 5863	23.3853 4966	23.9460 0743	24.5230 1460	25.1168 6844
20	24.2973 6980	24.9115 2003	25.5446 5761	26.1973 9750	26.8703 7449
21	25.7833 1719	26.4720 2023	27.1862 7405	27.9178 2593	28.6764 8572
22	27.2989 8354	28.0676 4989	28.8628 5590	29.6855 6615	30.5367 8030
23	28.8449 6321	29.6991 7201	30.5844 2730	31.5019 1921	32.4528 8370
24	30.4218 6247	31.3674 0338	32.3490 3798	33.3682 2199	34.4264 7022
25	32.0302 9972	33.0731 6996	34.1577 6393	35.2858 4810	36.4592 6432
26	33.6709 0572	34.8173 1628	36.0117 0803	37.2562 0892	38.5530 4225
27	35.3443 2383	36.6007 0590	37.9120 0073	39.2807 5467	40.7096 3352
28	37.0512 1031	38.4242 2178	39.8598 0075	41.3609 7542	42.9309 2252
29	38.7922 3451	40.2887 6677	41.8562 9577	43.4984 0224	45.2188 5020
30	40.5680 7921	42.1952 6402	43.9027 0316	45.6946 0830	47.5754 1571
31	42.3794 4079	44.1446 5746	46.0002 7074	47.9512 1003	50.0026 7818
32	44.2270 2961	46.1379 1226	48.1502 7751	50.2698 6831	52.5027 5852
33	46.1115 7020	48.1760 1528	50.3540 3445	52.6522 8969	55.0778 4128
34	48.0338 0160	50.2599 7563	52.6128 8531	55.1002 2765	57.7301 7652
35	49.9944 7763	52.3908 2508	54.9282 0744	57.6154 8391	60.4620 8181
36	51.9943 6719	54.5696 1864	57.3014 1263	60.1999 0972	63.2759 4427
37	54.0342 5453	56.7974 3506	59.7339 4794	62.8554 0724	66.1742 2259
38	56.1149 3962	59.0753 7735	62.2272 9664	65.5839 3094	69.1594 4927
39	58.2372 3841	61.4045 7334	64.7829 7906	68.3874 8904	72.2342 3275
40	60.4019 8318	63.7861 7624	67.4025 5354	71.2681 4499	75.4012 5973
41	62.6100 2284	66.2213 6521	70.0876 1737	74.2280 1898	78.6632 9753
42	64.8622 2330	68.7113 4592	72.8398 0781	77.2692 8950	82.0231 9645
43	67.1594 6777	71.2573 5121	75.6608 0300	80.3941 9496	85.4838 9234
44	69.5026 5712	73.8606 4161	78.5523 2308	83.6050 3532	89.0484 0911
45	71.8927 1027	76.5225 0605	81.5161 3116	86.9041 7379	92.7198 6139
46	74.3305 6447	79.2442 6243	84.5540 3443	90.2940 3857	96.5014 5723
47	76.8171 7576	82.0272 5834	87.6678 8530	93.7771 2463	100.3965 0095
48	79.3535 1927	84.8728 7165	90.8595 8243	97.3559 9556	104.4083 9598
49	81.9405 8966	87.7825 1126	94.1310 7199	101.0332 8544	108.5406 4785
50	84.5794 0145	90.7576 1776	97.4843 4879	104.8117 0079	112.7968 6729

TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

n	2%	$2\frac{1}{4}$%	$2\frac{1}{2}$%	$2\frac{3}{4}$%	3%
51	87.2709 8948	93.7996 6416	100.9214 5751	108.6940 2256	117.1807 7331
52	90.0164 0927	96.9101 5661	104.4444 9395	112.6831 0818	121.6961 9651
53	92.8167 3746	100.0906 3513	108.0556 0629	116.7818 9365	126.3470 8240
54	95.6730 7221	103.3426 7442	111.7569 9645	120.9933 9573	131.1374 9488
55	98.5865 3365	106.6678 8460	115.5509 2136	125.3207 1411	136.0716 1972
56	101.5582 6432	110.0679 1200	119.4396 9440	129.7670 3375	141.1537 6831
57	104.5894 2961	113.5444 4002	123.4256 8676	134.3356 2718	146.3883 8136
58	107.6812 1820	117.0991 8992	127.5113 2893	139.0298 5692	151.7800 3280
59	110.8348 4257	120.7339 2169	131.6991 1215	143.8531 7799	157.3334 3379
60	114.0515 3942	124.4504 3493	135.9915 8995	148.8091 4038	163.0534 3680
61	117.3325 7021	128.2505 6972	140.3913 7970	153.9013 9174	168.9450 3991
62	120.6792 2161	132.1362 0754	144.9011 6419	159.1336 8002	175.0133 9110
63	124.0928 0604	136.1092 7221	149.5236 9330	164.5098 5622	181.2637 9284
64	127.5746 6216	140.1717 3083	154.2617 8563	170.0338 7726	187.7017 0662
65	131.1261 5541	144.3255 9477	159.1183 3027	175.7098 0889	194.3327 5782
66	134.7486 7852	148.5729 2066	164.0962 8853	181.5418 2863	201.1627 4055
67	138.4436 5209	152.9158 1137	169.1986 9574	187.5342 2892	208.1976 2277
68	142.2125 2513	157.3564 1713	174.4286 6314	193.6914 2021	215.4435 5145
69	146.0567 7563	161.8969 3651	179.7893 7971	200.0179 3427	222.9068 5800
70	149.9779 1114	166.5396 1758	185.2841 1421	206.5184 2746	230.5940 6374
71	153.9774 6937	171.2867 5898	190.9162 1706	213.1976 8422	238.5118 8565
72	158.0570 1875	176.1407 1106	196.6891 2249	220.0606 2054	246.6672 4222
73	162.2181 5913	181.1038 7705	202.6063 5055	227.1122 8760	255.0672 5949
74	166.4625 2231	186.1787 1429	208.6715 0931	234.3578 7551	263.7192 7727
75	170.7917 7276	191.3677 3536	214.8882 9705	241.8027 1709	272.6308 5559
76	175.2076 0821	196.6735 0941	221.2605 0447	249.4522 9181	281.8097 8126
77	179.7117 6038	202.0986 6337	227.7920 1709	257.3122 2983	291.2640 7469
78	184.3059 9558	207.6458 8329	234.4868 1751	265.3883 1615	301.0019 9693
79	188.9921 1549	213.3179 1567	241.3489 8795	273.6864 9485	311.0320 5684
80	193.7719 5780	219.1175 6877	248.3827 1265	282.2128 7345	321.3630 1855
81	198.6473 9696	225.0477 1407	255.5922 8047	290.9737 2747	332.0039 0910
82	203.6203 4490	231.1112 8763	262.9820 8748	299.9755 0498	342.9640 2638
83	208.6927 5180	237.3112 9160	270.5566 3966	309.2248 3137	354.2529 4717
84	213.8666 0683	243.6507 9567	278.3205 5566	318.7285 1423	365.8805 3558
85	219.1439 3897	250.1329 3857	286.2785 6955	328.4935 4837	377.8569 5165
86	224.5268 1775	256.7609 2969	294.4355 3379	338.5271 2095	390.1926 6020
87	230.0173 5411	263.5380 5060	302.7964 2213	348.8366 1678	402.8984 4001
88	235.6177 0119	270.4676 5674	311.3663 3268	359.4296 2374	415.9853 9321
89	241.3300 5521	277.5531 7902	320.1504 9100	370.3139 3839	429.4649 5500
90	247.1566 5632	284.7981 2555	329.1542 5328	381.4975 7170	443.3489 0365
91	253.0997 8944	292.2060 8337	338.3831 0961	392.9887 5492	457.6493 7076
92	259.1617 8523	299.7807 2025	347.8426 8735	404.7959 4568	472.3788 5189
93	265.3450 2094	307.5257 8645	357.5387 5453	416.9278 3418	487.5502 1744
94	271.6519 2135	315.4451 1665	367.4772 2339	429.3933 4962	503.1767 2397
95	278.0849 5978	323.5426 3177	377.6641 5398	442.2016 6674	519.2720 2569
96	284.6466 5898	331.8223 4099	388.1057 5783	455.3622 1257	535.8501 8645
97	291.3395 9216	340.2883 4366	398.8084 0177	468.8846 7342	552.9256 9205
98	298.1663 8400	348.9448 3139	409.7786 1182	482.7790 0194	570.5134 6281
99	305.1297 1168	357.7960 9010	421.0230 7711	497.0554 2449	588.6288 6669
100	312.2323 0591	366.8465 0213	432.5486 5404	511.7244 4867	607.2877 3270

TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0350 0000	2.0400 0000	2.0450 0000	2.0500 0000	2.0550 0000
3	3.1062 2500	3.1216 0000	3.1370 2500	3.1525 0000	3.1680 2500
4	4.2149 4288	4.2464 6400	4.2781 9113	4.3101 2500	4.3422 6638
5	5.3624 6588	5.4163 2256	5.4707 0973	5.5256 3125	5.5810 9103
6	6.5501 5218	6.6329 7546	6.7168 9166	6.8019 1281	6.8880 5103
7	7.7794 0751	7.8992 9448	8.0191 5179	8.1420 0845	8.2668 9384
8	9.0516 8677	9.2142 2626	9.3800 1362	9.5491 0888	9.7215 7300
9	10.3684 9581	10.5827 9531	10.8021 1423	11.0265 6432	11.2562 5951
10	11.7313 9316	12.0061 0712	12.2882 0937	12.5778 9254	12.8753 5379
11	13.1419 9192	13.4863 5141	13.8411 7879	14.2067 8716	14.5834 9825
12	14.6019 6164	15.0258 0546	15.4650 3184	15.9171 2652	16.3855 9065
13	16.1130 3030	16.6268 3768	17.1599 1327	17.7129 8285	18.2867 9814
14	17.6769 8636	18.2919 1119	18.9321 0937	19.5986 3199	20.2925 7203
15	19.2956 8088	20.0235 8764	20.7840 5429	21.5785 6359	22.4086 6350
16	20.9710 2971	21.8245 3114	22.7193 3673	23.6574 9177	24.6411 3999
17	22.7050 1575	23.6975 1239	24.7417 0689	25.8403 6636	26.9964 0269
18	24.4996 9130	25.6454 1288	26.8550 8370	28.1323 8467	29.4812 0483
19	26.3571 8050	27.6712 2940	29.0635 6246	30.5390 0391	32.1026 7110
20	28.2796 8181	29.7780 7858	31.3714 2277	33.0659 5410	34.8683 1801
21	30.2694 7068	31.9692 0172	33.7831 3680	35.7192 5181	37.7860 7550
22	32.3289 0215	34.2479 6979	36.3033 7795	38.5052 1440	40.8643 0965
23	34.4604 1373	36.6178 8858	38.9370 2996	41.4304 7512	44.1118 4669
24	36.6665 2821	39.0826 0412	41.6891 9631	44.5019 9887	47.5379 9825
25	38.9498 5669	41.6459 0829	44.5652 1015	47.7270 9882	51.1525 8816
26	41.3131 0168	44.3117 4462	47.5706 4460	51.1134 5376	54.9659 8051
27	43.7590 6024	47.0842 1440	50.7113 2361	54.6691 2645	58.9891 0943
28	46.2906 2734	49.9675 8298	53.9933 3317	58.4025 8277	63.2335 1045
29	48.9107 9930	52.9662 8630	57.4230 3316	62.3227 1191	67.7113 5353
30	51.6226 7728	56.0849 3775	61.0070 6966	66.4388 4750	72.4354 7797
31	54.4294 7038	59.3283 3526	64.7523 8779	70.7607 8988	77.4194 2926
32	57.3345 0247	62.7014 6867	68.6662 4524	75.2988 2937	82.6774 9787
33	60.3412 1005	66.2095 2742	72.7562 2628	80.0637 7084	88.2247 6025
34	63.4531 5240	69.8579 0851	77.0302 5646	85.0669 5938	94.0771 2207
35	66.6740 1274	73.6522 2486	81.4966 1800	90.3203 0735	100.2513 6378
36	70.0076 0318	77.5983 1385	86.1639 6581	95.8363 2272	106.7651 8879
37	73.4578 6930	81.7022 4640	91.0413 4427	101.6281 3886	113.6372 7417
38	77.0288 9472	85.9703 3626	96.1382 0476	107.7095 4580	120.8873 2425
39	80.7249 0604	90.4091 4971	101.4644 2398	114.0950 2309	128.5361 2708
40	84.5502 7775	95.0255 1570	107.0303 2306	120.7997 7424	136.6056 1407
41	88.5095 3747	99.8265 3633	112.8466 8760	127.8397 6295	145.1189 2285
42	92.6073 7128	104.8195 9778	118.9247 8854	135.2317 5110	154.1004 6360
43	96.8486 2928	110.0123 8169	125.2764 0402	142.9933 3866	163.5759 8910
44	101.2383 3130	115.4128 7696	131.9138 4220	151.1430 0559	173.5726 6850
45	105.7816 7290	121.0293 9204	138.8499 6510	159.7001 5587	184.1191 6527
46	110.4840 3145	126.8705 6772	146.0982 1353	168.6851 6366	195.2457 1936
47	115.3509 7255	132.9453 9043	153.6726 3314	178.1194 2185	206.9842 3392
48	120.3882 5659	139.2632 0604	161.5879 0163	188.0253 9294	219.3683 6679
49	125.6018 4557	145.8337 3429	169.8593 5720	198.4266 6259	232.4336 2696
50	130.9979 1016	152.6670 8366	178.5030 2828	209.3479 9572	246.2174 7645

TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$
51	136.5828 3702	159.7737 6700	187.5356 6455	220.8153 9550	260.7594 3765
52	142.3632 3631	167.1647 1768	196.9747 6946	232.8561 6528	276.1012 0672
53	148.3459 4958	174.8513 0639	206.8386 3408	245.4989 7354	292.2867 7309
54	154.5380 5782	182.8453 5865	217.1463 7262	258.7739 2222	309.3625 4561
55	160.9468 8984	191.1591 7299	227.9179 5938	272.7126 1833	327.3774 8562
56	167.5800 3099	199.8055 3991	239.1742 6756	287.3482 4924	346.3832 4733
57	174.4453 3207	208.7977 6151	250.9371 0960	302.7156 6171	366.4343 2593
58	181.5509 1869	218.1496 7197	263.2292 7953	318.8514 4479	387.5882 1386
59	188.9052 0085	227.8756 5885	276.0745 9711	335.7940 1703	409.9055 6562
60	196.5168 8288	237.9906 8520	289.4979 5398	353.5837 1788	433.4503 7173
61	204.3949 7378	248.5103 1261	303.5253 6190	372.2629 0378	458.2901 4127
62	212.5487 9786	259.4507 2511	318.1840 0319	391.8760 4897	484.4960 9990
63	220.9880 0579	270.8287 5412	333.5022 8333	412.4698 5141	512.1433 8549
64	229.7225 8599	282.6619 0428	349.5098 8608	434.0933 4398	541.3112 7170
65	238.7628 7650	294.9683 8045	366.2378 3096	456.7980 1118	572.0833 9164
66	248.1195 7718	307.7671 1567	383.7185 3335	480.6379 1174	604.5479 7818
67	257.8037 6238	321.0778 0030	401.9858 6735	505.6698 0733	638.7981 1698
68	267.8268 9406	334.9209 1231	421.0752 3138	531.9532 9770	674.9320 1341
69	278.2008 3535	349.3177 4880	441.0236 1679	559.5509 6258	713.0532 7415
70	288.9378 6459	364.2904 5876	461.8696 7955	588.5285 1071	753.2712 0423
71	300.0506 8985	379.8620 7711	483.6538 1513	618.9549 3625	795.7011 2046
72	311.5524 6400	396.0565 6019	506.4182 3681	650.9026 8306	840.4646 8209
73	323.4568 0024	412.8988 2260	530.2070 5747	684.4478 1721	887.6902 3960
74	335.7777 8824	430.4147 7550	555.0663 7505	719.6702 0807	937.5132 0278
75	348.5300 1083	448.6313 6652	581.0443 6193	756.6537 1848	990.0764 2893
76	361.7285 6121	467.5766 2118	608.1913 5822	795.4864 0440	1045.5306 3252
77	375.3890 6085	487.2796 8603	636.5599 6934	836.2607 2462	1104.0348 1731
78	389.5276 7798	507.7708 7347	666.2051 6796	879.0737 6085	1165.7567 3226
79	404.1611 4671	529.0817 0841	697.1844 0052	924.0274 4889	1230.8733 5254
80	419.3067 8685	551.2449 7675	729.5576 9854	971.2288 2134	1299.5713 8693
81	434.9825 2439	574.2947 7582	763.3877 9497	1020.7902 6240	1372.0478 1321
82	451.2069 1274	598.2665 6685	798.7402 4575	1072.8297 7552	1448.5104 4294
83	467.9991 5469	623.1972 2952	835.6835 5680	1127.4712 6430	1529.1785 1730
84	485.3791 2510	649.1251 1870	874.2893 1686	1184.8448 2752	1614.2833 3575
85	503.3673 9448	676.0901 2345	914.6323 3612	1245.0870 6889	1704.0689 1921
86	521.9852 5329	704.1337 2839	956.7907 9125	1308.3414 2234	1798.7927 0977
87	541.2547 3715	733.2990 7753	1000.8463 7685	1374.7584 9345	1898.7263 0881
88	561.1986 5295	763.6310 4063	1046.8844 6381	1444.4964 1812	2004.1562 5579
89	581.8406 0581	795.1762 8225	1094.9942 6468	1517.7212 3903	2115.3848 4986
90	603.2050 2701	827.9833 3354	1145.2690 0659	1594.6073 0098	2232.7310 1660
91	625.3172 0295	862.1026 6688	1197.8061 1189	1675.3376 6603	2356.5312 2252
92	648.2033 0506	897.5867 7356	1252.7073 8692	1760.1045 4933	2487.1404 3976
93	671.8904 2073	934.4902 4450	1310.0792 1933	1849.1097 7680	2624.9331 6394
94	696.4065 8546	972.8698 5428	1370.0327 8420	1942.5652 6564	2770.3044 8796
95	721.7808 1595	1012.7846 4845	1432.6842 5949	2040.6935 2892	2923.6712 3480
96	748.0431 4451	1054.2960 3439	1498.1550 5117	2143.7282 0537	3085.4731 5271
97	775.2246 5457	1097.4678 7577	1566.5720 2847	2251.9146 1564	3256.1741 7611
98	803.3575 1748	1142.3665 9080	1638.0677 6976	2365.5103 4642	3436.2637 5580
99	832.4750 3059	1180.0612 5443	1712.7808 1939	2484.7858 6374	3626.2582 6237
100	862.6116 5666	1237.6237 0461	1790.8559 5627	2610.0251 5693	3826.7024 6680

TABLE V. The Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

n	6%	$6\frac{1}{2}$%	7%	$7\frac{1}{2}$%	8%
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000
2	2.0600 0000	2.0650 0000	2.0700 0000	2.0750 0000	2.0800 0000
3	3.1836 0000	3.1992 2500	3.2149 0000	3.2306 2500	3.2464 0000
4	4.3746 1600	4.4071 7463	4.4399 4300	4.4729 2188	4.5061 1200
5	5.6370 9296	5.6936 4098	5.7507 3901	5.8083 9102	5.8666 0096
6	6.9753 1854	7.0637 2764	7.1532 9074	7.2440 2034	7.3359 2904
7	8.3938 3765	8.5228 6994	8.6540 2109	8.7873 2187	8.9228 0336
8	9.8974 6791	10.0768 5648	10.2598 0257	10.4463 7101	10.6366 2763
9	11.4913 1598	11.7318 5215	11.9779 8875	12.2298 4883	12.4875 5784
10	13.1807 9494	13.4944 2254	13.8164 4796	14.1470 8750	14.4865 6247
11	14.9716 4264	15.3715 6001	15.7835 9932	16.2081 1906	16.6454 8746
12	16.8699 4120	17.3707 1141	17.8884 5127	18.4237 2799	18.9771 2646
13	18.8821 3767	19.4998 0765	20.1406 4286	20.8055 0759	21.4952 9658
14	21.0150 6593	21.7672 9515	22.5504 8786	23.3659 2066	24.2149 2030
15	23.2759 6988	24.1821 6933	25.1290 2201	26.1183 6470	27.1521 1393
16	25.6725 2808	26.7540 1034	27.8880 5355	29.0772 4206	30.3242 8304
17	28.2128 7976	29.4930 2101	30.8402 1730	32.2580 3521	33.7502 2569
18	30.9056 5255	32.4100 6738	33.9990 3251	35.6773 8785	37.4502 4374
19	33.7599 9170	35.5167 2176	37.3789 6479	39.3531 9194	41.4462 6324
20	36.7855 9120	38.8253 0867	40.9954 9232	43.3046 8134	45.7619 6430
21	39.9927 2668	42.3489 5373	44.8651 7678	47.5525 3244	50.4229 2144
22	43.3922 9028	46.1016 3573	49.0057 3916	52.1189 7237	55.4567 5516
23	46.9958 2769	50.0982 4205	53.4361 4090	57.0278 9530	60.8932 9557
24	50.8155 7735	54.3546 2778	58.1766 7076	62.3049 8744	66.7647 5922
25	54.8645 1200	58.8876 7859	63.2490 3772	67.9778 6150	73.1059 3995
26	59.1563 8272	63.7153 7769	68.6764 7036	74.0762 0112	79.9544 1515
27	63.7057 6568	68.8568 7725	74.4838 2328	80.6319 1620	87.3507 6836
28	68.5281 1162	74.3325 7427	80.6976 9091	87.6793 0991	95.3388 2983
29	73.6397 9832	80.1641 9159	87.3465 2927	95.2552 5816	103.9659 3622
30	79.0581 8622	86.3748 6405	94.4607 8632	103.3994 0252	113.2832 1111
31	84.8016 7739	92.9892 3021	102.0730 4137	112.1543 5771	123.3458 6800
32	90.8897 7803	100.0335 3017	110.2181 5426	121.5659 3454	134.2135 3744
33	97.3431 6471	107.5357 0963	118.9334 2506	131.6833 7963	145.9506 2044
34	104.1837 5460	115.5255 3076	128.2587 6481	142.5596 3310	158.6266 7007
35	111.4347 7987	124.0346 9026	138.2368 7835	154.2516 0558	172.3168 0368
36	119.1208 6666	133.0969 4513	148.9134 5984	166.8204 7600	187.1021 4797
37	127.2681 1866	142.7482 4656	160.3374 0202	180.3320 1170	203.0703 1981
38	135.9042 0578	153.0268 8259	172.5610 2017	194.8569 1258	220.3159 4540
39	145.0584 5813	163.9736 2995	185.6402 9158	210.4711 8102	238.9412 2103
40	154.7619 6562	175.6319 1590	199.6351 1199	227.2565 1960	259.0565 1871
41	165.0476 8356	188.0479 9044	214.6095 6983	245.3007 5857	280.7810 4021
42	175.9505 4457	201.2711 0981	230.6322 3972	264.6983 1546	304.2435 2342
43	187.5075 7724	215.3537 3195	247.7764 9650	285.5506 8912	329.5830 0530
44	199.7580 3188	230.3517 2453	266.1208 5125	307.9669 9080	356.9496 4572
45	212.7435 1379	246.3245 8662	285.7493 1084	332.0645 1511	386.5056 1738
46	226.5081 2462	263.3356 8475	306.7517 6260	357.9693 5375	418.4260 6677
47	241.0986 1210	281.4525 0426	329.2243 8598	385.8170 5528	452.9001 5211
48	256.5645 2882	300.7469 1704	353.2700 9300	415.7533 3442	490.1321 6428
49	272.9584 0055	321.2954 6665	378.9989 9951	447.9348 3451	530.3427 3742
50	290.3359 0458	343.1796 7198	406.5289 2947	482.5299 4709	573.7701 5642

TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n|}\,i} = \frac{1-(1+i)^{-n}}{i}$$

n	$\frac{1}{8}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{8}\%$	1%
1	0.9966 7774	0.9958 5062	0.9950 2488	0.9913 2590	0.9900 9901
2	1.9900 4426	1.9875 6908	1.9850 9938	1.9740 5294	1.9703 9506
3	2.9801 1056	2.9751 7253	2.9702 4814	2.9482 5570	2.9409 8521
4	3.9668 8760	3.9586 7804	3.9504 9566	3.9140 0813	3.9019 6555
5	4.9503 8631	4.9381 0261	4.9258 6633	4.8713 8352	4.8534 3124
6	5.9306 1759	5.9134 6318	5.8963 8441	5.8204 5454	5.7954 7647
7	6.9075 9228	6.8847 7661	6.8620 7404	6.7612 9323	6.7281 9453
8	7.8813 2121	7.8520 5969	7.8229 5924	7.6939 7098	7.6516 7775
9	8.8518 1516	8.8153 2915	8.7790 6392	8.6185 5859	8.5660 1758
10	9.8190 8487	9.7746 0164	9.7304 1186	9.5351 2624	9.4713 0453
11	10.7831 4107	10.7298 9374	10.6770 2673	10.4437 4348	10.3676 2825
12	11.7439 9442	11.6812 2198	11.6189 3207	11.3444 7929	11.2550 7747
13	12.7016 5557	12.6286 0280	12.5561 5131	12.2374 0202	12.1337 4007
14	13.6561 3512	13.5720 5257	13.4887 0777	13.1225 7945	13.0037 0304
15	14.6074 4364	14.5115 8762	14.4166 2465	14.0000 7876	13.8650 5252
16	15.5555 9167	15.4472 2418	15.3399 2502	14.8690 6656	14.7178 7378
17	16.5005 8970	16.3789 7843	16.2586 3186	15.7323 0885	15.5622 5127
18	17.4424 4821	17.3068 6648	17.1727 6802	16.5871 7111	16.3982 6858
19	18.3811 7762	18.2309 0438	18.0823 5624	17.4346 1820	17.2260 0850
20	19.3167 8832	19.1511 0809	18.9874 1915	18.2747 1445	18.0455 5297
21	20.2492 9069	20.0674 9352	19.8879 7925	19.1075 2361	18.8569 8313
22	21.1786 9504	20.9800 7653	20.7840 5896	19.9331 0891	19.6603 7934
23	22.1050 1167	21.8888 7289	21.6756 8055	20.7515 3300	20.4558 2113
24	23.0282 5083	22.7938 9831	22.5628 6622	21.5628 5799	21.2433 8726
25	23.9484 2275	23.6951 6843	23.4456 3803	22.3671 4547	22.0231 5570
26	24.8655 3763	24.5926 9884	24.3240 1794	23.1644 5647	22.7952 0366
27	25.7796 0561	25.4865 0506	25.1980 2780	23.8548 5152	23.5596 0759
28	26.6906 3682	26.3766 0254	26.0676 8936	24.7383 9060	24.3164 4316
29	27.5986 4135	27.2630 0668	26.9330 2423	25.5151 3319	25.0657 8530
30	28.5036 2925	28.1457 3278	27.7940 5397	26.2851 3823	25.8077 0822
31	29.4056 1055	29.0247 9612	28.6507 9997	27.0484 6417	26.5422 8537
32	30.3045 9523	29.9002 1189	29.5032 8355	27.8051 6894	27.2695 8947
33	31.2005 9325	30.7719 9524	30.3515 2592	28.5553 0998	27.9896 9255
34	32.0936 1454	31.6401 6122	31.1955 4818	29.2989 4422	28.7026 6589
35	32.9836 6898	32.5047 2486	32.0353 7132	30.0361 2809	29.4085 8009
36	33.8707 6642	33.3657 0109	32.8710 1624	30.7669 1757	30.1075 0504
37	34.7549 1670	34.2231 0481	33.7025 0372	31.4913 6810	30.7995 0994
38	35.6361 2960	35.0769 5084	34.5298 5445	32.2095 3467	31.4846 6330
39	36.5144 1488	35.9272 5394	35.3530 8900	32.9214 7179	32.1630 3298
40	37.3897 8228	36.7740 2881	36.1722 2786	33.6272 3350	32.8346 8611
41	38.2622 4147	37.6172 9009	36.9872 9141	34.3268 7335	33.4996 8922
42	39.1318 0213	38.4570 5236	37.7982 9991	35.0204 4446	34.1581 0814
43	39.9984 7388	39.2933 3013	38.6052 7354	35.7079 9947	34.8100 0806
44	40.8622 6633	40.1261 3788	39.4082 3238	36.3895 9055	35.4554 5352
45	41.7231 8903	40.9554 8999	40.2071 9640	37.0652 6944	36.0945 0844
46	42.5812 5153	41.7814 0081	41.0021 8547	37.7350 8743	36.7272 3608
47	43.4364 6332	42.6038 8461	41.7932 1937	38.3990 9535	37.3536 9909
48	44.2888 3387	43.4229 5562	42.5803 1778	39.0573 4359	37.9739 5949
49	45.1383 7263	44.2386 2799	43.3635 0028	39.7098 8212	38.5880 7871
50	45.9850 2900	45.0509 1582	44.1427 8635	40.3567 6047	39.1961 1753

TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{8}\%$	1%
51	46.8289 9236	45.8598 3317	44.9181 9537	40.9980 2772	39.7981 3617
52	47.6700 9205	46.6653 9401	45.6897 4664	41.6337 3256	40.3941 9423
53	48.5083 9739	47.4676 1228	46.4574 5934	42.2639 2324	40.9843 5072
54	49.3439 1767	48.2665 0184	47.2213 5258	42.8886 4757	41.5686 6408
55	50.1766 6213	49.0620 7651	47.9814 4535	43.5079 5298	42.1471 9216
56	51.0066 3999	49.8543 5003	48.7377 5657	44.1218 8647	42.7199 9224
57	51.8338 6046	50.6433 3612	49.4903 0505	44.7304 9465	43.2871 2102
58	52.6583 3268	51.4290 4840	50.2391 0950	45.3338 2369	43.8486 3468
59	53.4800 6580	52.2115 0046	50.9841 8855	45.9519 1939	44.4045 8879
60	54.2990 6890	52.9907 0584	51.7255 6075	46.5248 2716	44.9550 3841
61	55.1153 5106	53.7666 7800	52.4632 4453	47.1125 9198	45.5000 3803
62	55.9289 2133	54.5394 3035	53.1972 5824	47.6952 5847	46.0396 4161
63	56.7397 8870	55.3089 7627	53.9276 2014	48.2728 7085	46.5739 0258
64	57.5479 6216	56.0753 2905	54.6543 4839	48.8454 7296	47.1028 7385
65	58.3534 5065	56.8385 0194	55.3774 6109	49.4131 0826	47.6266 0777
66	59.1562 6311	57.5985 0814	56.0969 7621	49.9758 1984	48.1451 5621
67	59.9564 0842	58.3553 6078	56.8129 1165	50.5336 5040	48.6585 7050
68	60.7538 9543	59.1090 7296	57.5252 8522	51.0866 4228	49.1669 0149
69	61.5487 3299	59.8596 5770	58.2341 1465	51.6348 3745	49.6701 9949
70	62.3409 2989	60.6071 2798	58.9394 1756	52.1782 7752	50.1685 1435
71	63.1304 9490	61.3514 9672	59.6412 1151	52.7170 0374	50.6618 9539
72	63.9174 3678	62.0927 7680	60.3395 1394	53.2510 5699	51.1503 9148
73	64.7017 6423	62.8309 8103	61.0343 4222	53.7804 7781	51.6340 5097
74	65.4834 8595	63.5661 2216	61.7257 1366	54.3053 0638	52.1129 2175
75	66.2626 1058	64.2982 1292	62.4136 4543	54.8255 8253	52.5870 5124
76	67.0391 4676	65.0272 6596	63.0981 5466	55.3413 4575	53.0564 8637
77	67.8131 0308	65.7532 9388	63.7792 5836	55.8526 3520	53.5212 7364
78	68.5844 8812	66.4763 0924	64.4569 7350	56.3594 8966	53.9814 5905
79	69.3533 1042	67.1963 2453	65.1313 1691	56.8619 4762	54.4370 8817
80	70.1195 7849	67.9133 5221	65.8023 0538	57.3600 4721	54.8882 0611
81	70.8833 0082	68.6274 0467	66.4699 5561	57.8538 2623	55.3348 5753
82	71.6444 8587	69.3384 9426	67.1342 8419	58.3433 2216	55.7770 8666
83	72.4031 4206	70.0466 3326	67.7953 0765	58.8285 7215	56.2149 3729
84	73.1592 7780	70.7518 3393	68.4530 4244	59.3096 1304	56.6484 5276
85	73.9129 0146	71.4541 0846	69.1075 0491	59.7864 8133	57.0776 7600
86	74.6640 2139	72.1534 6898	69.7587 1135	60.2592 1321	57.5026 4951
87	75.4126 4591	72.8499 2759	70.4066 7796	60.7278 4457	57.9234 1535
88	76.1587 8329	73.5434 9633	71.0514 2086	61.1924 1097	58.3400 1520
89	76.9024 4182	74.2341 8720	71.6929 5608	61.6529 4768	58.7524 9030
90	77.6436 2972	74.9220 1212	72.3312 9958	62.1094 8965	59.1608 8148
91	78.3823 5520	75.6069 8300	72.9664 6725	62.5620 7152	59.5652 2919
92	79.1186 2645	76.2891 1168	73.5984 7487	63.0107 2765	59.9655 7346
93	79.8524 5161	76.9684 0995	74.2273 3818	63.4554 9210	60.3619 5392
94	80.5838 3882	77.6448 8955	74.8530 7282	63.8963 9861	60.7544 0982
95	81.3127 9616	78.3185 6218	75.4756 9434	64.3334 8065	61.1429 8002
96	82.0393 3172	78.9894 3950	76.0952 1825	64.7667 7140	61.5277 0299
97	82.7634 5354	79.6575 3308	76.7116 5995	65.1963 0375	61.9086 1682
98	83.4851 6964	80.3228 5450	77.3250 3478	65.6221 1028	62.2857 5923
99	84.2044 8802	80.9854 1524	77.9353 5799	66.0442 2333	62.6591 6755
100	84.9214 1663	81.6452 2677	78.5426 4477	66.4626 7492	63.0288 7877

TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
1	0.9888 7515	0.9876 5432	0.9864 3650	0.9852 2167	0.9828 0098
2	1.9667 4923	1.9631 1538	1.9594 9346	1.9558 8342	1.9486 9875
3	2.9337 4460	2.9265 3371	2.9193 5237	2.9122 0042	2.8979 8403
4	3.8899 8230	3.8780 5798	3.8661 9222	3.8543 8465	3.8309 4254
5	4.8355 8200	4.8178 3504	4.8001 8962	4.7826 4497	4.7478 5508
6	5.7706 6205	5.7460 0992	5.7215 1874	5.6971 8717	5.6489 9762
7	6.6953 3948	6.6627 2585	6.6303 5140	6.5982 1396	6.5346 4130
8	7.6097 3002	7.5681 2429	7.5268 5712	7.4859 2508	7.4050 5297
9	8.5139 4810	8.4623 4498	8.4112 0308	8.3605 1732	8.2604 9432
10	9.4081 0690	9.3455 2591	9.2835 5421	9.2221 8455	9.1012 2291
11	10.2923 1832	10.2178 0337	10.1440 7320	10.0711 1779	9.9274 9181
12	11.1666 9302	11.0793 1197	10.9929 2054	10.9075 0521	10.7395 4969
13	12.0313 4044	11.9301 8466	11.8302 5454	11.7315 3222	11.5376 4097
14	12.8863 6880	12.7705 5275	12.6562 3136	12.5433 8150	12.3220 0587
15	13.7318 8509	13.6005 4592	13.4710 0504	13.3432 3301	13.0928 8046
16	14.5679 9514	14.4202 9227	14.2747 2754	14.1312 6405	13.8504 9677
17	15.3948 0360	15.2299 1829	15.0675 4874	14.9076 4931	14.5950 8282
18	16.2124 1395	16.0295 4893	15.8496 1651	15.6725 6089	15.3268 6272
19	17.0209 2850	16.8193 0759	16.6210 7671	16.4261 6837	16.0460 5673
20	17.8204 4845	17.5993 1613	17.3820 7320	17.1686 3879	16.7528 8130
21	18.6110 7387	18.3696 9495	18.1327 4792	17.9001 3673	17.4475 4919
22	19.3929 0371	19.1305 6291	18.8732 4086	18.6208 2437	18.1302 6948
23	20.1660 3580	19.8820 3744	19.6036 9012	19.3308 6145	18.8012 4764
24	20.9305 6693	20.6242 3451	20.3242 3193	20.0304 0537	19.4606 8565
25	21.6865 9276	21.3572 6865	21.0350 0067	20.7196 1120	20.1087 8196
26	22.4342 0792	22.0812 5299	21.7361 2890	21.3986 3172	20.7457 3166
27	23.1735 0598	22.7962 9925	22.4277 4737	22.0676 1746	21.3717 2644
28	23.9045 7946	23.5025 1778	23.1099 8508	22.7267 1671	21.9869 5474
29	24.6275 1986	24.2000 1756	23.7829 6925	23.3760 7558	22.5916 0171
30	25.3424 1766	24.8889 0623	24.4468 2540	24.0158 3801	23.1858 4934
31	26.0493 6233	25.5692 9010	25.1016 7734	24.6461 4582	23.7698 7650
32	26.7484 4236	26.2412 7418	25.7476 4719	25.2671 3874	24.3438 5897
33	27.4397 4522	26.9049 6215	26.3848 5543	25.8789 5442	24.9079 6951
34	28.1233 5745	27.5604 5644	27.0134 2089	26.4817 2849	25.4623 7789
35	28.7993 6460	28.2078 5822	27.6334 6080	27.0755 9458	26.0072 6100
36	29.4678 5127	28.8472 6737	28.2450 9080	27.6606 8431	26.5427 5283
37	30.1289 0114	29.4787 8259	28.8484 2496	28.2371 2740	27.0690 4455
38	30.7825 9692	30.1025 0133	29.4435 7579	28.8050 5163	27.5862 8457
39	31.4230 2044	30.7185 1983	30.0306 5430	29.3645 8288	28.0946 2857
40	32.0682 5260	31.3269 3316	30.6097 6996	29.9158 4520	28.5942 2955
41	32.7903 7340	31.9278 3522	31.1810 3079	30.4589 6079	29.0852 3789
42	33.3254 6195	32.5213 1874	31.7445 4332	30.9940 5004	29.5678 0135
43	33.9435 9649	33.1074 7530	32.3004 1264	31.5212 3157	30.0420 6522
44	34.5548 5438	33.6863 9536	32.8487 4243	32.0406 2223	30.5081 7221
45	35.1593 1212	34.2581 6825	33.3896 3495	32.5523 3718	30.9662 6261
46	35.7570 4536	34.8228 8222	33.9231 9108	33.0564 8983	31.4164 7431
47	36.3481 2891	35.3806 2442	34.4495 1031	33.5531 9195	31.8589 4281
48	36.9326 3674	35.9314 8091	34.9686 9081	34.0425 5365	32.2938 0129
49	37.5106 4202	36.4755 3670	35.4808 2941	34.5246 8339	32.7211 8063
50	38.0822 1708	37.0128 7574	35.9860 2161	34.9996 8807	33.1412 0946

TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
51	38.6474 3345	37.5435 8099	36.4843 6164	35.4676 7298	33.5540 1421
52	39.2063 6188	38.0677 3431	36.9759 4243	35.9287 4185	33.9597 1913
53	39.7590 7232	38.5854 1660	37.4608 5566	36.3829 9690	34.3584 4633
54	40.3056 3394	39.0967 0776	37.9391 9178	36.8305 3882	34.7503 1579
55	40.8461 1514	39.6016 8667	38.4110 3998	37.2714 6681	35.1354 4550
56	41.3805 8358	40.1004 3128	38.8764 8826	37.7058 7863	35.5139 5135
57	41.9091 0613	40.5930 1855	39.3356 2344	38.1338 7058	35.8859 4727
58	42.4317 4896	41.0795 2449	39.7885 3114	38.5555 3751	36.2515 4523
59	42.9485 7746	41.5600 2419	40.2352 9582	38.9709 7292	36.6108 5526
60	43.4596 5633	42.0345 9179	40.6760 0081	39.3802 6889	36.9639 8552
61	43.9650 4952	42.5033 0054	41.1107 2829	39.7835 1614	37.3110 4228
62	44.4648 2029	42.9662 2275	41.5395 5935	40.1808 0408	37.6521 3000
63	44.9590 3119	43.4234 2988	41.9625 7396	40.5722 2077	37.9873 5135
64	45.4477 4407	43.8749 0217	42.3798 5101	40.9578 5298	38.3168 0723
65	45.9310 2009	44.3209 8022	42.7914 6832	41.3377 8618	38.6405 9678
66	46.4089 1975	44.7614 6195	43.1975 0266	41.7121 0461	38.9588 1748
67	46.8815 0284	45.1965 0563	43.5989 2075	42.0808 9125	39.2715 6509
68	47.3488 2852	45.6261 7840	43.9931 2429	42.4442 2783	39.5789 3375
69	47.8109 5527	46.0505 4656	44.3828 5997	42.8021 9490	39.8810 1597
70	48.2679 4094	46.4696 7502	44.7673 0946	43.1548 7183	40.1779 0267
71	48.7198 4270	46.8836 3024	45.1465 4448	43.5023 3678	40.4696 8321
72	49.1667 1714	47.2924 7431	45.5206 3573	43.8446 6677	40.7564 4542
73	49.6086 2016	47.6962 7093	45.8896 5300	44.1819 3771	41.0382 7560
74	50.0456 0708	48.0950 8240	46.2536 6511	44.5142 2434	41.3152 5857
75	50.4777 3259	48.4889 7027	46.6127 3994	44.8416 0034	41.5874 7771
76	50.9050 5077	48.8779 9533	46.9669 4445	45.1641 3826	41.8550 1495
77	51.3276 1510	49.2622 1761	47.3163 4471	45.4819 0962	42.1179 5081
78	51.7454 7847	49.6416 9640	47.6610 0588	45.7949 8485	42.3763 6443
79	52.1586 9317	50.0164 9027	48.0009 9224	46.1034 3335	42.6303 3359
80	52.5673 1092	50.3866 5706	48.3363 6719	46.4073 2349	42.8799 3474
81	52.9713 8286	50.7522 5389	48.6671 9328	46.7067 2265	43.1252 4298
82	53.3709 5957	51.1133 3717	48.9935 3221	47.0016 9720	43.3663 3217
83	53.7660 9104	51.4699 6264	49.3154 4484	47.2923 1251	43.6032 7486
84	54.1568 2674	51.8221 8532	49.6329 9122	47.5786 3301	43.8361 4237
85	54.5432 1557	52.1700 5958	49.9462 3055	47.8607 2218	44.0650 0479
86	54.9253 0588	52.5136 3909	50.2552 2125	48.1386 4254	44.2899 3099
87	55.3031 4549	52.8529 7688	50.5600 2096	48.4124 5571	44.5109 8869
88	55.6767 8169	53.1881 2531	50.8606 8653	48.6822 2237	44.7282 4441
89	56.0462 6126	53.5191 3611	51.1572 7401	48.9480 0234	44.9417 6355
90	56.4116 3041	53.8460 6035	51.4498 3873	49.2098 5452	45.1516 1037
91	56.7729 3490	54.1689 4850	51.7384 3524	49.4678 3696	45.3578 4803
92	57.1302 1992	54.4878 5037	52.0231 1738	49.7220 0686	45.5605 3860
93	57.4835 3021	54.8028 1518	52.3039 3823	49.9724 2055	45.7597 4310
94	57.8329 0997	55.1138 9154	52.5809 5016	50.2191 3355	45.9555 2147
95	58.1784 0294	55.4211 2744	52.8542 0484	50.4622 0054	46.1479 3265
96	58.5200 5235	55.7245 7031	53.1237 5324	50.7016 7541	46.3370 3455
97	58.8579 0096	56.0242 6698	53.3896 4561	50.9376 1124	46.5228 8408
98	59.1919 9106	56.3202 6368	53.6519 3155	51.1700 6034	46.7055 3718
99	59.5223 6446	56.6126 0610	53.9106 5998	51.3990 7422	46.8850 4882
100	59.8490 6251	56.9013 3936	54.1658 7914	51.6247 0367	47.0614 7304

TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

n	2%	$2\frac{1}{4}$%	$2\frac{1}{2}$%	$2\frac{3}{4}$%	3%
1	0.9803 9216	0.9779 9511	0.9756 0976	0.9732 3601	0.9708 7379
2	1.9415 6094	1.9344 6955	1.9274 2415	1.9204 2434	1.9134 6970
3	2.8838 8327	2.8698 9687	2.8560 2356	2.8422 6213	2.8286 1135
4	3.8077 2870	3.7847 4021	3.7619 7421	3.7394 2787	3.7170 9840
5	4.7134 5951	4.6794 5253	4.6458 2850	4.6125 8186	4.5797 0719
6	5.6014 3089	5.5544 7680	5.5081 2536	5.4623 6678	5.4171 9144
7	6.4719 9107	6.4102 4626	6.3493 9060	6.2894 0806	6.2302 8296
8	7.3254 8144	7.2471 8461	7.1701 3717	7.0943 1441	7.0196 9219
9	8.1622 3671	8.0657 0622	7.9708 6553	7.8776 7826	7.7861 0892
10	8.9825 8501	8.8662 1635	8.7520 6393	8.6400 7616	8.5302 0284
11	9.7868 4805	9.6491 1134	9.5142 0871	9.3820 6926	9.2526 2411
12	10.5753 4122	10.4147 7882	10.2577 6460	10.1042 0366	9.9540 0399
13	11.3483 7375	11.1635 9787	10.9831 8497	10.8070 1086	10.6349 5533
14	12.1062 4877	11.8959 3924	11.6909 1217	11.4910 0814	11.2960 7314
15	12.8492 6350	12.6121 6551	12.3813 7773	12.1566 9892	11.9379 3509
16	13.5777 0931	13.3126 3131	13.0550 0266	12.8045 7315	12.5611 0203
17	14.2918 7188	13.9976 8343	13.7121 9772	13.4351 0769	13.1661 1847
18	14.9920 3125	14.6676 6106	14.3533 6363	14.0487 6661	13.7535 1308
19	15.6784 6201	15.3228 9590	14.9788 9134	14.6460 0157	14.3237 9911
20	16.3514 3334	15.9637 1237	15.5891 6229	15.2272 5213	14.8774 7486
21	17.0112 0916	16.5904 2775	16.1845 4857	15.7929 4612	15.4150 2414
22	17.6580 4820	17.2033 5232	16.7654 1324	16.3434 9987	15.9369 1664
23	18.2922 0412	17.8027 8955	17.3321 1048	16.8793 1861	16.4436 0839
24	18.9139 2560	18.3890 3624	17.8849 8583	17.4007 9670	16.9355 4212
25	19.5234 5647	18.9623 8263	18.4243 7642	17.9083 1795	17.4131 4769
26	20.1210 3576	19.5231 1280	18.9506 1114	18.4022 5592	17.8768 4242
27	20.7068 9780	20.0715 0376	19.4640 1087	18.8829 7413	18.3270 3147
28	21.2812 7236	20.6078 2764	19.9648 8866	19.3508 2640	18.7641 0823
29	21.8443 8466	21.1323 4977	20.4535 4991	19.8061 5708	19.1884 5459
30	22.3964 5555	21.6453 2985	20.9302 9259	20.2493 0130	19.6004 4135
31	22.9377 0152	22.1470 2186	21.3954 0741	20.6805 8520	20.0004 2849
32	23.4683 3482	22.6376 7419	21.8491 7796	21.1003 2623	20.3887 6553
33	23.9885 6355	23.1175 2977	22.2918 8094	21.5088 3332	20.7657 9178
34	24.4985 9172	23.5868 2618	22.7237 8628	21.9064 0712	21.1318 3668
35	24.9986 1933	24.0457 9577	23.1451 5734	22.2933 4026	21.4872 2007
36	25.4888 4248	24.4946 6579	23.5562 5107	22.6699 1753	21.8322 5250
37	25.9694 5341	24.9336 5848	23.9573 1812	23.0364 1609	22.1672 3544
38	26.4406 4060	25.3629 9118	24.3486 0304	23.3931 0568	22.4924 6159
39	26.9025 8883	25.7828 7646	24.7303 4443	23.7402 4884	22.8082 1513
40	27.3554 7924	26.1935 2221	25.1027 7505	24.0781 0106	23.1147 7197
41	27.7994 8945	26.5951 3174	25.4661 2200	24.4069 1101	23.4123 9997
42	28.2347 9358	26.9879 0390	25.8206 0683	24.7269 2069	23.7013 5920
43	28.6615 6233	27.3720 3316	26.1664 4569	25.0383 6563	23.9819 0213
44	29.0799 6307	27.7477 0969	26.5038 4945	25.3414 7507	24.2542 7392
45	29.4901 5987	28.1151 1950	26.8330 2386	25.6364 7209	24.5187 1254
46	29.8923 1360	28.4744 4450	27.1541 6962	25.9235 7381	24.7754 4907
47	30.2865 8196	28.8258 6259	27.4674 8255	26.2029 9154	25.0247 0783
48	30.6731 1957	29.1695 4777	27.7731 5371	26.4749 3094	25.2667 0664
49	31.0520 7801	29.5056 7019	28.0713 6947	26.7395 9215	25.5016 5693
50	31.4236 0589	29.8343 9627	28.3623 1168	26.9971 6998	25.7297 6401

TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

n	2%	$2\frac{1}{4}$%	$2\frac{1}{2}$%	$2\frac{3}{4}$%	3%
51	31.7878 4892	30.1558 8877	28.6461 5774	27.2478 5400	25.9512 2719
52	32.1449 4992	30.4703 0687	28.9230 8072	27.4918 2871	26.1662 3999
53	32.4950 4894	30.7778 0623	29.1932 4948	27.7292 7368	26.3749 9028
54	32.8382 8327	31.0785 3910	29.4568 2876	27.9603 6368	26.5776 6047
55	33.1747 8752	31.3726 5438	29.7139 7928	28.1852 6879	26.7744 2764
56	33.5046 9365	31.6602 9768	29.9648 5784	28.4041 5454	26.9654 6373
57	33.8281 3103	31.9416 1142	30.2096 1740	28.6171 8203	27.1509 3566
58	34.1452 2650	32.2167 3489	30.4484 0722	28.8245 0806	27.3310 0549
59	34.4561 0441	32.4858 0429	30.6813 7290	29.0262 8522	27.5058 3058
60	34.7608 8668	32.7489 5285	30.9086 5649	29.2226 6201	27.6755 6367
61	35.0596 9282	33.0063 1086	31.1303 9657	29.4137 8298	27.8403 5307
62	35.3526 4002	33.2580 0573	31.3467 2836	29.5997 8879	28.0003 4279
63	35.6398 4316	33.5041 6208	31.5577 8377	29.7808 1634	28.1556 7261
64	35.9214 1486	33.7449 0179	31.7636 9148	29.9569 9887	28.3064 7826
63	36.1974 6555	33.9803 4405	31.9645 7705	30.1284 6605	28.4528 9152
66	36.4681 0348	34.2106 0543	32.1605 6298	30.2953 4409	28.5950 4031
67	36.7334 3478	34.4357 9993	32.3517 6876	30.4577 5581	28.7330 4884
68	36.9935 6351	34.6560 3905	32.5383 1099	30.6158 2074	28.8670 3771
69	37.2485 9168	34.8714 3183	32.7203 0340	30.7696 5522	28.9971 2399
70	37.4986 1920	35.0820 8492	32.8978 5698	30.9193 7247	29.1234 2135
71	37.7437 4441	35.2881 0261	33.0710 7998	31.0650 8270	29.2460 4015
72	37.9840 6314	35.4895 8691	33.2400 7803	31.2068 9314	29.3650 8752
73	38.2196 6975	35.6866 3756	33.4049 5417	31.3449 0816	29.4806 6750
74	38.4506 5662	35.8793 5214	33.5658 0895	31.4792 2936	29.5928 8106
75	38.6771 1433	36.0678 2605	33.7227 4044	31.6099 5558	29.7018 2628
76	38.8991 3170	36.2521 5262	33.8758 4433	31.7371 8304	29.8075 9833
77	39.1167 9578	36.4324 2310	34.0252 1398	31.8610 0540	29.9102 8964
78	39.3301 9194	36.6087 2675	34.1709 4047	31.9815 1377	30.0099 8994
79	39.5394 0386	36.7811 5085	34.3131 1265	32.0987 9685	30.1067 8635
80	39.7445 1359	36.9497 8079	34.4518 1722	32.2129 4098	30.2007 6345
81	39.9456 0156	37.1147 0004	34.5871 3875	32.3240 3015	30.2920 0335
82	40.1427 4663	37.2759 9026	34.7191 5976	32.4321 4613	30.3805 8577
83	40.3360 2611	37.4337 3130	34.8479 6074	32.5373 6850	30.4665 8813
84	40.5255 1579	37.5880 0127	34.9736 2023	32.6397 7469	30.5500 8556
85	40.7112 8999	37.7388 7655	35.0962 1486	32.7394 4009	30.6311 5103
86	40.8934 2156	37.8864 3183	35.2158 1938	32.8364 3804	30.7098 5537
87	41.0719 8192	38.0307 4018	35.3325 0671	32.9308 3994	30.7862 6735
88	41.2470 4110	38.1718 7304	35.4463 4801	33.0227 1527	30.8604 5374
89	41.4186 6774	38.3099 0028	35.5574 1269	33.1121 3165	30.9324 7936
90	41.5869 2916	38.4448 9025	35.6657 6848	33.1991 5489	31.0024 0714
91	41.7518 9133	38.5769 0978	35.7714 8144	33.2838 4905	31.0702 9820
92	41.9136 1985	38.7060 2423	35.8746 1604	33.3662 7644	31.1362 1184
93	42.0721 7545	38.8322 9754	35.9752 3516	33.4464 9776	31.2002 0567
94	42.2276 2299	38.9557 9221	36.0734 0016	33.5245 7202	31.2623 3560
95	42.3800 2254	39.0765 6940	36.1691 7089	33.6005 5671	31.3226 5592
96	42.5294 3386	39.1946 8890	36.2626 0574	33.6745 0775	31.3812 1934
97	42.6759 1555	39.3102 0920	36.3537 6170	33.7464 7956	31.4380 7703
98	42.8195 2505	39.4231 8748	36.4426 9434	33.8165 2512	31.4932 7867
99	42.9603 1867	39.5336 7968	36.5294 5790	33.8846 9598	31.5468 7250
100	43.0983 5164	39.6417 4052	36.6141 0526	33.9510 4232	31.5989 0534

TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$
1	0.9661 8357	0.9615 3846	0.9569 3780	0.9523 8095	0.9478 6730
2	1.8996 9428	1.8860 9467	1.8726 6775	1.8594 1043	1.8463 1971
3	2.8016 3698	2.7750 9103	2.7489 6435	2.7232 4803	2.6979 3338
4	3.6730 7921	3.6298 9522	3.5875 2570	3.5459 5050	3.5051 5012
5	4.5150 5238	4.4518 2233	4.3899 7674	4.3294 7667	4.2702 8448
6	5.3285 5302	5.2421 3686	5.1578 7248	5.0756 9206	4.9955 3031
7	6.1145 4298	6.0020 5467	5.8927 0094	5.7863 7340	5.6829 6712
8	6.8739 5554	6.7327 4487	6.5958 8607	6.4632 1276	6.3345 6599
9	7.6076 8651	7.4353 3161	7.2687 9050	7.1078 2168	6.9521 9525
10	8.3166 0532	8.1108 9578	7.9127 1818	7.7217 3493	7.5376 2583
11	9.0015 5104	8.7604 7671	8.5289 1692	8.3064 1422	8.0925 3633
12	9.6633 3433	9.3850 7376	9.1185 8078	8.8632 5164	8.6185 1785
13	10.3027 3849	9.9856 4785	9.6828 5242	9.3935 7299	9.1170 7853
14	10.9205 2028	10.5631 2293	10.2228 2528	9.8986 4094	9.5896 4790
15	11.5174 1090	11.1183 8743	10.7395 4573	10.3796 5804	10.0375 8094
16	12.0941 1681	11.6522 9561	11.2340 1505	10.8377 6956	10.4621 6203
17	12.6513 2059	12.1656 6885	11.7071 9143	11.2740 6625	10.8646 0856
18	13.1896 8173	12.6592 9697	12.1599 9180	11.6895 8690	11.2460 7447
19	13.7098 3742	13.1339 3940	12.5932 9359	12.0853 2086	11.6076 5352
20	14.2124 0330	13.5903 2634	13.0079 3645	12.4622 1034	11.9503 8249
21	14.6979 7420	14.0291 5995	13.4047 2388	12.8211 5271	12.2752 4406
22	15.1671 2484	14.4511 1533	13.7844 2476	13.1630 0258	12.5831 6973
23	15.6204 1047	14.8568 4167	14.1477 7489	13.4885 7388	12.8750 4240
24	16.0583 6760	15.2469 6314	14.4954 7837	13.7986 4179	13.1516 9825
25	16.4815 1459	15.6220 7994	14.8282 0896	14.0939 4457	13.4139 3266
26	16.8903 5226	15.9827 6918	15.1466 1145	14.3751 8530	13.6624 9541
27	17.2853 6451	16.3295 8575	15.4513 0282	14.6430 3362	13.8980 9991
28	17.6670 1885	16.6630 6322	15.7428 7351	14.8981 2726	14.1214 2172
29	18.0357 6700	16.9837 1463	16.0218 8853	15.1410 7358	14.3331 0116
30	18.3920 4541	17.2920 3330	16.2888 8854	15.3724 5103	14.5337 4517
31	18.7362 7576	17.5884 9356	16.5443 9095	15.5928 1050	14.7239 2907
32	19.0688 6547	17.8735 5150	16.7888 9086	15.8026 7667	14.9041 9817
33	19.3902 0818	18.1467 4567	17.0228 6207	16.0025 4921	15.0750 6936
34	19.7006 8423	18.4111 9776	17.2467 5796	16.1929 0401	15.2370 3257
35	20.0006 6110	18.6646 1323	17.4610 1240	16.3741 9429	15.3905 5220
36	20.2904 9381	18.9082 8195	17.6660 4058	16.5468 5171	15.5360 6843
37	20.5705 2542	19.1425 7880	17.8622 3979	16.7112 8734	15.6739 9851
38	20.8410 8736	19.3678 6423	18.0499 9023	16.8678 9271	15.8047 3793
39	21.1024 9987	19.5844 8484	18.2296 5572	17.0170 4067	15.9286 6154
40	21.3550 7234	19.7927 7388	18.4015 8442	17.1590 8635	16.0461 2469
41	21.5991 0371	19.9930 5181	18.5661 0949	17.2943 6796	16.1574 6416
42	21.8348 8281	20.1856 2674	18.7235 4975	17.4232 0758	16.2629 9920
43	22.0626 8870	20.3707 9494	18.8742 1029	17.5459 1198	16.3630 3242
44	22.2827 9102	20.5488 4129	19.0183 8305	17.6627 7331	16.4578 5063
45	22.4954 5026	20.7200 3970	19.1563 4742	17.7740 6982	16.5477 2572
46	22.7009 1813	20.8846 5356	19.2883 7074	17.8800 6650	16.6329 1537
47	22.8994 3780	21.0429 3612	19.4147 0884	17.9810 1571	16.7136 6386
48	23.0912 4425	21.1951 3088	19.5356 0654	18.0771 5782	16.7902 0271
49	23.2765 6450	21.3414 7200	19.6512 9813	18.1687 2173	16.8627 5139
50	23.4556 1757	21.4821 8462	19.7620 0778	18.2559 2546	16.9315 1790

TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$
51	23.6286 1630	21.6174 8521	19.8679 5003	18.3389 7663	16.9966 9943
52	23.7957 6454	21.7475 8193	19.9693 3017	18.4180 7298	17.0584 8287
53	23.9572 6043	21.8726 7493	20.0663 4466	18.4934 0284	17.1170 4538
54	24.1132 9510	21.9929 5667	20.1591 8149	18.5651 4556	17.1725 5486
55	24.2640 5323	22.1086 1218	20.2480 2057	18.6334 7196	17.2251 7048
56	24.4097 1327	22.2189 1940	20.3330 3404	18.6985 4473	17.2750 4311
57	24.5504 4760	22.3267 4943	20.4143 8664	18.7605 1879	17.3223 1575
58	24.6864 2281	22.4295 6676	20.4922 3602	18.8195 4170	17.3671 2393
59	24.8177 9981	22.5284 2957	20.5667 3303	18.8757 5400	17.4095 9614
60	24.9447 3412	22.6234 8997	20.6380 2204	18.9292 8952	17.4498 5416
61	25.0673 7596	22.7148 9421	20.7062 4118	18.9802 7574	17.4880 1343
62	25.1858 7049	22.8027 8289	20.7715 2266	19.0288 3404	17.5241 8334
63	25.3003 5796	22.8872 9124	20.8339 9298	19.0750 8003	17.5584 6762
64	25.4109 7388	22.9685 4927	20.8937 7319	19.1191 2384	17.5909 6457
65	25.5178 4916	23.0466 8199	20.9509 7913	19.1610 7033	17.6217 6737
66	25.6211 1030	23.1218 0961	21.0057 2165	19.2010 1936	17.6509 6433
67	25.7208 7951	23.1940 4770	21.0581 0684	19.2390 6606	17.6786 3917
68	25.8172 7480	23.2635 0740	21.1082 3621	19.2753 0101	17.7048 7125
69	25.9104 1052	23.3302 9558	21.1562 0690	19.3098 1048	17.7297 3579
70	26.0003 9664	23.3945 1498	21.2021 1187	19.3426 7665	17.7533 0406
71	26.0873 3975	23.4562 6440	21.2460 4007	19.3739 7776	17.7756 4366
72	26.1713 4275	23.5156 3885	21.2880 7662	19.4037 8834	17.7968 1864
73	26.2525 0508	23.5727 2966	21.3283 0298	19.4321 7937	17.8168 8970
74	26.3309 2278	23.6276 2468	21.3667 9711	19.4592 1845	17.8359 1441
75	26.4066 8868	23.6804 0834	21.4036 3360	19.4849 6995	17.8539 4731
76	26.4798 9244	23.7311 6187	21.4388 8383	19.5094 9519	17.8710 4010
77	26.5506 2072	23.7799 6333	21.4726 1611	19.5328 5257	17.8872 4180
78	26.6189 5721	23.8268 8782	21.5048 9579	19.5550 9768	17.9025 9887
79	26.6849 8281	23.8720 0752	21.5357 8545	19.5762 8351	17.9171 5532
80	26.7487 7567	23.9153 9185	21.5653 4493	19.5964 6048	17.9309 5291
81	26.8104 1127	23.9571 0754	21.5936 3151	19.6156 7665	17.9440 3120
82	26.8699 6258	23.9972 1879	21.6207 0001	19.6339 7776	17.8564 2768
83	26.9275 0008	24.0357 8730	21.6466 0288	19.6514 0739	17.9681 7789
84	26.9830 9186	24.0728 7240	21.6713 9032	19.6680 0704	17.9793 1554
85	27.0368 0373	24.1085 3116	21.6951 1035	19.6838 1623	17.9898 7255
86	27.0886 9926	24.1428 1842	21.7178 0895	19.6988 7260	17.9998 7919
87	27.1388 3986	24.1757 8694	21.7395 3009	19.7132 1200	18.0093 6416
88	27.1872 8489	24.2074 8745	21.7603 1588	19.7268 6857	18.0183 5466
89	27.2340 9168	24.2379 6870	21.7802 0658	19.7398 7483	18.0268 7645
90	27.2793 1564	24.2672 7759	21.7992 4075	19.7522 6174	18.0349 5398
91	27.3230 1028	24.2954 5923	21.8174 5526	19.7640 5880	18.0426 1041
92	27.3652 2732	24.3225 5695	21.8348 8542	19.7752 9410	18.0498 6769
93	27.4060 1673	24.3486 1245	21.8515 6499	19.7859 9438	18.0567 4662
94	27.4454 2680	24.3736 6582	21.8675 2631	19.7961 8512	18.0632 6694
95	27.4835 0415	24.3977 5559	21.8828 0030	19.8058 9059	18.0694 4734
96	27.5202 9387	24.4209 1884	21.8974 1655	19.8151 3390	18.0753 0553
97	27.5558 3948	24.4431 9119	21.9114 0340	19.8239 3705	18.0808 5833
98	27.5901 8308	24.4646 0692	21.9247 8794	19.8323 2100	18.0861 2164
99	27.6233 6529	24.4851 9896	21.9375 9612	19.8403 0571	18.0911 1055
100	27.6554 2540	24.5049 9900	21.9498 5274	19.8479 1020	18.0958 3939

TABLE VI. The Present Value of 1 per Annum at Compound Interest

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

n	6%	$6\frac{1}{2}$%	7%	$7\frac{1}{2}$%	8%
1	0.9433 9623	0.9389 6714	0.9345 7944	0.9302 3256	0.9259 2593
2	1.8333 9267	1.8206 2642	1.8080 1817	1.7955 6517	1.7832 6475
3	2.6730 1195	2.6484 7551	2.6243 1604	2.6005 2574	2.5770 9699
4	3.4651 0561	3.4257 9860	3.3872 1126	3.3493 2627	3.3121 2684
5	4.2123 6379	4.1556 7944	4.1001 9744	4.0458 8490	3.9927 1004
6	4.9173 2433	4.8410 1356	4.7665 3966	4.6938 4642	4.6228 7966
7	5.5823 8144	5.4845 1977	5.3892 8940	5.2966 0132	5.2063 7006
8	6.2097 9381	6.0887 5096	5.9712 9851	5.8573 0355	5.7466 3894
9	6.8016 9227	6.6561 0419	6.5152 3225	6.3788 8703	6.2468 8791
10	7.3600 8705	7.1888 3022	7.0235 8154	6.8640 8096	6.7100 8140
11	7.8868 7458	7.6890 4246	7.4986 7434	7.3154 2415	7.1389 6426
12	8.3838 4394	8.1587 2532	7.9426 8630	7.7352 7827	7.5360 7802
13	8.8526 8296	8.5997 4208	8.3576 5074	8.1258 4026	7.9037 7594
14	9.2949 8393	9.0138 4233	8.7454 6799	8.4891 5373	8.2442 3698
15	9.7122 4899	9.4026 6885	9.1079 1401	8.8271 1974	8.5594 7809
16	10.1058 9527	9.7677 6418	9.4466 4860	9.1415 0674	8.8513 6916
17	10.4772 5969	10.1105 7670	9.7632 2299	9.4339 5976	9.1216 3811
18	10.8276 0348	10.4324 6638	10.0590 8691	9.7060 0908	9.3718 8714
19	11.1581 1649	10.7347 1022	10.3355 9524	9.9590 7821	9.6035 9920
20	11.4699 2122	11.0185 0725	10.5940 1425	10.1944 9136	9.8181 4741
21	11.7640 7662	11.2849 8333	10.8355 2733	10.4134 8033	10.0168 0316
22	12.0415 8172	11.5351 9562	11.0612 4050	10.6171 9101	10.2007 4366
23	12.3033 7898	11.7701 3673	11.2721 8738	10.8066 8931	10.3710 5895
24	12.5503 5753	11.9907 3871	11.4693 3400	10.9829 6680	10.5287 5828
25	12.7833 5616	12.1978 7672	11.6535 8318	11.1469 4586	10.6747 7619
26	13.0031 6619	12.3923 7251	11.8257 7867	11.2994 8452	10.8099 7795
27	13.2105 3414	12.5749 9766	11.9867 0904	11.4413 8095	10.9351 6477
28	13.4061 6428	12.7464 7668	12.1371 1125	11.5733 7763	11.0510 7849
29	13.5907 2102	12.9074 8984	12.2776 7407	11.6961 6524	11.1584 0601
30	13.7648 3115	13.0586 7591	12.4090 4118	11.8103 8627	11.2577 8334
31	13.9290 8599	13.2006 3465	12.5318 1419	11.9166 3839	11.3497 9939
32	14.0840 4339	13.3339 2925	12.6465 5532	12.0154 7757	11.4349 9944
33	14.2302 2961	13.4590 8850	12.7537 9002	12.1074 2099	11.5138 8837
34	14.3681 4114	13.5766 0802	12.8540 0936	12.1929 4976	11.5869 3367
35	14.4982 4636	13.6869 5673	12.9476 7230	12.2725 1141	11.6545 6822
36	14.6209 8713	13.7905 6970	13.0352 0776	12.3465 2224	11.7171 9279
37	14.7367 8031	13.8878 5887	13.1170 1660	12.4153 6953	11.7751 7851
38	14.8460 1916	13.9792 1021	13.1934 7345	12.4794 1351	11.8288 6899
39	14.9490 7468	14.0649 8611	13.2649 2846	12.5389 8931	11.8785 8240
40	15.0462 9687	14.1455 2687	13.3317 0884	12.5944 0866	11.9246 1333
41	15.1380 1592	14.2211 5199	13.3941 2041	12.6459 6155	11.9672 3457
42	15.2245 4332	14.2921 6149	13.4524 4898	12.6939 1772	12.0066 9867
43	15.3061 7294	14.3588 3708	13.5069 6167	12.7385 2811	12.0432 3951
44	15.3831 8202	14.4214 4327	13.5579 0810	12.7800 2615	12.0770 7362
45	15.4558 3209	14.4802 2842	13.6055 2159	12.8186 2898	12.1084 0150
46	15.5243 6990	14.5354 2575	13.6500 2018	12.8545 3858	12.1374 0880
47	15.5890 2821	14.5872 5422	13.6916 0764	12.8879 4287	12.1642 6741
48	15.6500 2661	14.6359 1946	13.7304 7443	12.9190 1662	12.1891 3649
49	15.7075 7227	14.6816 1451	13.7667 9853	12.9479 2244	12.2121 6341
50	15.7618 6064	14.7245 2067	13.8007 4629	12.9748 1157	12.2334 8464

TABLE VII. The Annuity Whose Present Value at Compound Interest is 1

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n}|i}}$$

n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{8}\%$	1%
1	1.0033 3333	1.0041 6667	1.0050 0000	1.0087 5000	1.0100 0000
2	0.5025 0139	0.5031 2717	0.5037 5312	0.5065 7203	0.5075 1244
3	0.3355 5802	0.3361 1496	0.3366 7221	0.3391 8361	0.3400 2211
4	0.2520 8680	0.2526 0958	0.2531 3279	0.2554 9257	0.2562 8109
5	0.2020 0444	0.2025 0693	0.2030 0997	0.2052 8049	0.2060 3980
6	0.1686 1650	0.1691 0564	0.1695 9546	0.1718 0789	0.1725 4837
7	0.1447 6824	0.1452 4800	0.1457 2854	0.1479 0070	0.1486 2828
8	0.1268 8228	0.1273 5512	0.1278 2886	0.1299 7190	0.1306 9029
9	0.1129 7118	0.1134 3876	0.1139 0736	0.1160 2868	0.1167 4037
10	0.1018 4248	0.1023 0596	0.1027 7057	0.1048 7538	0.1055 8208
11	0.0927 3736	0.0931 9757	0.0936 5903	0.0957 5111	0.0964 5408
12	0.0851 4990	0.0856 0748	0.0860 6643	0.0881 4860	0.0888 4879
13	0.0787 2989	0.0791 8532	0.0796 4224	0.0817 1669	0.0824 1482
14	0.0732 2716	0.0736 8082	0.0741 3609	0.0762 0453	0.0769 0117
15	0.0684 0777	0.0689 1045	0.0693 6436	0.0714 2817	0.0721 2378
16	0.0642 8557	0.0647 3655	0.0651 8937	0.0672 4965	0.0679 4460
17	0.0606 0389	0.0610 5387	0.0615 0579	0.0635 6346	0.0642 5806
18	0.0573 3140	0.0577 8053	0.0582 3173	0.0602 8756	0.0609 8205
19	0.0544 0348	0.0548 5191	0.0553 0253	0.0573 5715	0.0580 5175
20	0.0517 6844	0.0522 1630	0.0526 6645	0.0547 2042	0.0554 1532
21	0.0493 8445	0.0498 3183	0.0502 8163	0.0523 3541	0.0530 3075
22	0.0472 1726	0.0476 6427	0.0481 1380	0.0501 6779	0.0508 6371
23	0.0452 3861	0.0456 8531	0.0461 3465	0.0481 8921	0.0488 8584
24	0.0434 2492	0.0438 7139	0.0443 2061	0.0463 7604	0.0470 7347
25	0.0417 5640	0.0422 0270	0.0426 5186	0.0447 0843	0.0454 0675
26	0.0402 1630	0.0406 6247	0.0411 1163	0.0431 6959	0.0438 6888
27	0.0387 9035	0.0392 3645	0.0396 8565	0.0417 4520	0.0424 4553
28	0.0374 6632	0.0379 1239	0.0383 6167	0.0404 2300	0.0411 2444
29	0.0362 3367	0.0366 7974	0.0371 2914	0.0391 9243	0.0398 9502
30	0.0350 8325	0.0355 2936	0.0359 7892	0.0380 4431	0.0387 4811
31	0.0340 0712	0.0344 5330	0.0349 0304	0.0369 7068	0.0376 7573
32	0.0329 9830	0.0334 4458	0.0338 9453	0.0359 6454	0.0366 7089
33	0.0320 5067	0.0324 9708	0.0329 4727	0.0350 1976	0.0357 2744
34	0.0311 5885	0.0316 0540	0.0320 5586	0.0341 3092	0.0348 3997
35	0.0303 1803	0.0307 6476	0.0312 1550	0.0332 9324	0.0340 0368
36	0.0295 2399	0.0299 7090	0.0304 2194	0.0325 0244	0.0332 1431
37	0.0287 7291	0.0292 2003	0.0296 7139	0.0317 5473	0.0324 6805
38	0.0280 6141	0.0285 0875	0.0289 6045	0.0310 4671	0.0317 6150
39	0.0273 8644	0.0278 3402	0.0282 8607	0.0303 7531	0.0310 9160
40	0.0267 4527	0.0271 9310	0.0276 4552	0.0297 3780	0.0304 5560
41	0.0261 3543	0.0263 8352	0.0270 3631	0.0291 3169	0.0298 5102
42	0.0255 5466	0.0260 0303	0.0264 5622	0.0285 5475	0.0292 7563
43	0.0250 0095	0.0254 4961	0.0259 0320	0.0280 0493	0.0287 2737
44	0.0244 7246	0.0249 2141	0.0253 7541	0.0274 8039	0.0282 0441
45	0.0239 6749	0.0244 1675	0.0248 7117	0.0269 7943	0.0277 0505
46	0.0234 8451	0.0239 3409	0.0243 8894	0.0265 0053	0.0272 2775
47	0.0230 2213	0.0234 7204	0.0239 2733	0.0260 4228	0.0267 7111
48	0.0225 7905	0.0230 2929	0.0234 8503	0.0256 0338	0.0263 3384
49	0.0221 5410	0.0226 0468	0.0230 6087	0.0251 0265	0.0259 1474
50	0.0217 4618	0.0221 9711	0.0226 5376	0.0247 7900	0.0255 1273

TABLE VII. The Annuity Whose Present Value at Compound Interest is 1

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n}|i}}$$

n	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{8}\%$	1%
51	0.0213 5429	0.0218 0557	0.0222 6269	0.0243 9142	0.0251 2680
52	0.0209 7751	0.0214 2916	0.0218 8675	0.0240 1899	0.0247 5603
53	0.0206 1499	0.0210 6700	0.0215 2507	0.0236 6084	0.0243 9956
54	0.0202 6592	0.0207 1830	0.0211 7686	0.0233 1619	0.0240 5658
55	0.0199 2958	0.0203 8234	0.0208 4139	0.0229 8430	0.0237 2637
56	0.0196 0529	0.0200 5843	0.0205 1797	0.0226 6449	0.0234 0823
57	0.0192 9241	0.0197 4593	0.0202 0598	0.0223 5611	0.0231 0156
58	0.0189 9035	0.0194 4426	0.0199 0481	0.0220 5858	0.0228 0573
59	0.0186 9856	0.0191 5287	0.0196 1392	0.0217 7135	0.0225 2020
60	0.0184 1652	0.0188 7123	0.0193 3280	0.0214 9390	0.0222 4445
61	0.0181 4377	0.0185 9888	0.0190 6096	0.0212 2575	0.0219 7800
62	0.0178 7984	0.0183 3536	0.0187 9796	0.0209 6644	0.0217 2041
63	0.0176 2432	0.0180 8025	0.0185 4337	0.0207 1557	0.0214 7125
64	0.0173 7681	0.0178 3315	0.0182 9681	0.0204 7273	0.0212 3013
65	0.0171 3695	0.0175 9371	0.0180 5789	0.0202 3754	0.0209 9667
66	0.0169 0438	0.0173 6156	0.0178 2627	0.0200 0968	0.0207 7052
67	0.0166 7878	0.0171 3639	0.0176 0163	0.0197 8879	0.0205 5136
68	0.0164 5985	0.0169 1788	0.0173 8366	0.0195 7459	0.0203 3888
69	0.0162 4729	0.0167 0574	0.0171 7206	0.0193 6677	0.0201 3280
70	0.0160 4083	0.0164 9971	0.0169 6657	0.0191 6506	0.0199 3282
71	0.0158 4021	0.0162 9952	0.0167 6693	0.0189 6921	0.0197 3870
72	0.0156 4518	0.0161 0493	0.0165 7289	0.0187 7897	0.0195 5019
73	0.0154 5553	0.0159 1572	0.0163 8422	0.0185 9411	0.0193 6706
74	0.0152 7103	0.0157 3165	0.0162 0070	0.0184 1441	0.0191 8910
75	0.0150 9147	0.0155 5253	0.0160 2214	0.0182 3966	0.0190 1609
76	0.0149 1666	0.0153 7816	0.0158 4832	0.0180 6967	0.0188 4784
77	0.0147 4641	0.0152 0836	0.0156 7908	0.0179 0426	0.0186 8416
78	0.0145 8056	0.0150 4295	0.0155 1423	0.0177 4324	0.0185 2488
79	0.0144 1892	0.0148 8177	0.0153 5360	0.0175 8645	0.0183 6984
80	0.0142 6135	0.0147 2464	0.0151 9704	0.0174 3374	0.0182 1885
81	0.0141 0770	0.0145 7144	0.0150 4439	0.0172 8494	0.0180 7180
82	0.0139 5781	0.0144 2200	0.0148 9552	0.0171 3992	0.0179 2851
83	0.0138 1156	0.0142 7620	0.0147 5028	0.0169 9854	0.0177 8886
84	0.0136 6881	0.0141 3301	0.0146 0855	0.0168 6067	0.0176 5273
85	0.0135 2944	0.0139 9500	0.0144 7021	0.0167 2619	0.0175 1998
86	0.0133 9333	0.0138 5935	0.0143 3513	0.0165 9497	0.0173 9050
87	0.0132 6038	0.0137 2685	0.0142 0320	0.0164 6691	0.0172 6417
89	0.0131 3046	0.0135 9740	0.0140 7431	0.0163 4190	0.0171 4089
89	0.0130 0349	0.0134 7088	0.0139 4837	0.0162 1982	0.0170 2056
90	0.0128 7936	0.0133 4721	0.0138 2527	0.0161 0060	0.0169 0306
91	0.0127 5797	0.0132 2629	0.0137 0493	0.0159 8413	0.0167 8832
92	0.0126 3925	0.0131 0803	0.0135 8724	0.0158 7031	0.0166 7624
93	0.0125 2310	0.0129 9234	0.0134 7213	0.0157 5908	0.0165 6673
94	0.0124 0944	0.0128 7915	0.0133 5950	0.0156 5033	0.0164 5971
95	0.0122 9819	0.0127 6837	0.0132 4930	0.0155 4401	0.0163 5511
96	0.0121 8928	0.0126 5992	0.0131 4143	0.0154 4002	0.0162 5284
97	0.0120 8263	0.0125 5374	0.0130 3583	0.0153 3829	0.0161 5284
98	0.0119 7818	0.0124 4976	0.0129 3242	0.0152 3877	0.0160 5503
99	0.0118 7585	0.0123 4790	0.0128 3115	0.0151 4137	0.0159 5936
100	0.0117 7559	0.0122 4811	0.0127 3194	0.0150 4604	0.0158 6574

TABLE VII. The Annuity Whose Present Value at Compound Interest is

$$\frac{1}{a_{\overline{n|}i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n|}i}}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
1	1.0112 5000	1.0125 0000	1.0137 5000	1.0150 0000	1.0175 0000
2	0.5084 5323	0.5093 9441	0.5103 3597	0.5112 7792	0.5131 6295
3	0.3408 6130	0.3417 0117	0.3425 4173	0.3433 8296	0.3450 6746
4	0.2570 7058	0.2578 6102	0.2586 5243	0.2594 4478	0.2610 3237
5	0.2068 0034	0.2075 6211	0.2083 2510	0.2090 8932	0.2106 2142
6	0.1732 9034	0.1740 3381	0.1747 7877	0.1755 2521	0.1770 2256
7	0.1493 5762	0.1500 8872	0.1508 2157	0.1515 5616	0.1530 3059
8	0.1314 1071	0.1321 3314	0.1328 5758	0.1335 8402	0.1350 4292
9	0.1174 5432	0.1181 7055	0.1188 8906	0.1196 0982	0.1210 5813
10	0.1062 9131	0.1070 0307	0.1077 1737	0.1084 3418	0.1098 7534
11	0.0971 5984	0.0978 6839	0.0985 7973	0.0992 9384	0.1007 3038
12	0.0895 5203	0.0902 5831	0.0909 6764	0.0916 7999	0.0931 1377
13	0.0831 1626	0.0838 2100	0.0845 2303	0.0852 4036	0.0866 7283
14	0.0776 0138	0.0783 0515	0.0790 1246	0.0797 2332	0.0811 5562
15	0.0728 2321	0.0735 2646	0.0742 3351	0.0749 4436	0.0763 7739
16	0.0686 4363	0.0693 4672	0.0700 5388	0.0707 6508	0.0721 9958
17	0.0649 5698	0.0656 6023	0.0663 6730	0.0670 7966	0.0685 1623
18	0.0616 8113	0.0623 8479	0.0630 9301	0.0638 0578	0.0652 4492
19	0.0587 5120	0.0594 5548	0.0601 6457	0.0608 7847	0.0623 2061
20	0.0561 1531	0.0568 2039	0.0575 3054	0.0582 4574	0.0596 9122
21	0.0537 3145	0.0544 3748	0.0551 4881	0.0558 6550	0.0573 1464
22	0.0515 6525	0.0522 7238	0.0529 8507	0.0537 0331	0.0551 5638
23	0.0495 8833	0.0502 9666	0.0510 1080	0.0517 3075	0.0531 8796
24	0.0477 7701	0.0484 8665	0.0492 0235	0.0499 2410	0.0513 8565
25	0.0461 1144	0.0468 2247	0.0475 3981	0.0482 6345	0.0497 2952
26	0.0445 7479	0.0452 8729	0.0460 0635	0.0467 3196	0.0482 0269
27	0.0431 5273	0.0438 6677	0.0445 8763	0.0453 1527	0.0467 9079
28	0.0418 3299	0.0425 4863	0.0432 7134	0.0440 0108	0.0454 8151
29	0.0406 0498	0.0413 2228	0.0420 4689	0.0427 7878	0.0442 6424
30	0.0394 5953	0.0401 7854	0.0409 0511	0.0416 3919	0.0431 2975
31	0.0383 8866	0.0391 0942	0.0398 3798	0.0405 7430	0.0420 7005
32	0.0373 8535	0.0381 0791	0.0388 3850	0.0395 7710	0.0410 7812
33	0.0364 4349	0.0371 6786	0.0379 0053	0.0386 4144	0.0401 4779
34	0.0355 5763	0.0362 8387	0.0370 1864	0.0377 6189	0.0392 7363
35	0.0347 2299	0.0354 5111	0.0361 8801	0.0369 3363	0.0384 5082
36	0.0339 3529	0.0346 6533	0.0354 0438	0.0361 5240	0.0376 7507
37	0.0331 9072	0.0339 2270	0.0346 6394	0.0354 1437	0.0369 4257
38	0.0324 8589	0.0332 1983	0.0339 6327	0.0347 1613	0.0362 4990
39	0.0318 1773	0.0325 5365	0.0332 9931	0.0340 5463	0.0355 9399
40	0.0311 8349	0.0319 2141	0.0326 6931	0.0334 2710	0.0349 7209
41	0.0305 8069	0.0313 2063	0.0320 7078	0.0328 3106	0.0343 8170
42	0.0300 0709	0.0307 4906	0.0315 0148	0.0322 6426	0.0338 2057
43	0.0294 6064	0.0302 0466	0.0309 5936	0.0317 2465	0.0332 8666
44	0.0289 3949	0.0296 8557	0.0304 4257	0.0312 1038	0.0327 7810
45	0.0284 4197	0.0291 9012	0.0299 4941	0.0307 1976	0.0322 9321
46	0.0279 6652	0.0287 1675	0.0294 7836	0.0302 5125	0.0318 3043
47	0.0275 1173	0.0282 6406	0.0290 2799	0.0298 0342	0.0313 8836
48	0.0270 7632	0.0278 3075	0.0285 9701	0.0293 7500	0.0309 6569
49	0.0266 5910	0.0274 1563	0.0281 8424	0.0289 6478	0.0305 6124
50	0.0262 5898	0.0270 1763	0.0277 8857	0.0285 7168	0.0301 7391

TABLE VII. The Annuity Whose Present Value at Compound Interest is 1

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n}|i}}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
51	0.0258 7494	0.0266 3571	0.0274 0900	0.0281 9469	0.0298 0269
52	0.0255 0606	0.0262 6897	0.0270 4461	0.0278 3287	0.0294 4665
53	0.0251 5149	0.0259 1653	0.0266 9453	0.0274 8537	0.0291 0492
54	0.0248 1043	0.0255 7760	0.0263 5797	0.0271 5138	0.0287 7672
55	0.0244 8213	0.0252 5145	0.0260 3418	0.0268 3018	0.0284 6129
56	0.0241 6592	0.0249 3739	0.0257 2249	0.0265 2106	0.0281 5795
57	0.0238 6116	0.0246 3478	0.0254 2225	0.0262 2341	0.0278 6606
58	0.0235 6726	0.0243 4303	0.0251 3287	0.0259 3661	0.0275 8503
59	0.0232 8366	0.0240 6158	0.0248 5380	0.0256 6012	0.0273 1430
60	0.0230 0985	0.0237 8993	0.0245 8452	0.0253 9343	0.0270 5336
61	0.0227 4534	0.0235 2758	0.0243 2455	0.0251 3604	0.0268 0172
62	0.0224 8969	0.0232 7410	0.0240 7344	0.0248 8751	0.0265 5892
63	0.0222 4247	0.0230 2904	0.0238 3076	0.0246 4741	0.0263 2455
44	0.0220 0329	0.0227 9203	0.0235 9612	0.0244 1534	0.0260 9821
65	0.0217 7178	0.0225 6268	0.0233 6914	0.0241 9094	0.0258 7952
66	0.0215 4758	0.0223 4065	0.0231 4949	0.0239 7386	0.0256 6813
67	0.0213 3037	0.0221 2560	0.0229 3682	0.0237 6376	0.0254 6372
68	0.0211 1985	0.0219 1724	0.0227 3082	0.0235 6033	0.0252 6596
69	0.0209 1571	0.0217 1527	0.0225 3122	0.0233 6329	0.0250 7459
70	0.0207 1769	0.0215 1941	0.0223 3773	0.0231 7235	0.0248 8930
71	0.0205 2552	0.0213 2941	0.0221 5009	0.0229 8727	0.0247 0985
72	0.0203 3896	0.0211 4501	0.0219 6806	0.0228 0779	0.0245 3600
73	0.0201 5779	0.0209 6600	0.0217 9140	0.0226 3368	0.0243 6750
74	0.0199 8177	0.0207 9215	0.0216 1991	0.0224 6473	0.0242 0413
75	0.0198 1072	0.0206 2325	0.0214 5336	0.0223 0072	0.0240 4570
76	0.0196 4442	0.0204 5910	0.0212 9157	0.0221 4146	0.0238 9200
77	0.0194 8269	0.0202 9953	0.0211 3435	0.0219 8676	0.0237 4284
78	0.0193 2536	0.0201 4435	0.0209 8151	0.0218 3645	0.0235 9806
79	0.0191 7226	0.0199 9341	0.0208 3290	0.0216 9036	0.0234 5748
80	0.0190 2323	0.0198 4652	0.0206 8836	0.0215 4832	0.0233 2093
81	0.0188 7812	0.0197 0356	0.0205 4772	0.0214 1019	0.0231 8828
82	0.0187 3678	0.0195 6437	0.0204 1086	0.0212 7583	0.0230 5936
83	0.0185 9908	0.0194 2881	0.0202 7762	0.0211 4509	0.0229 3406
84	0.0184 6489	0.0192 9675	0.0201 4789	0.0210 1784	0.0228 1223
85	0.0183 3409	0.0191 6808	0.0200 2153	0.0208 9396	0.0226 9375
86	0.0182 0654	0.0190 4267	0.0198 9843	0.0207 7333	0.0225 7850
87	0.0180 8215	0.0189 2041	0.0197 7847	0.0206 5584	0.0224 6636
88	0.0179 6081	0.0188 0119	0.0196 6155	0.0205 4138	0.0223 5724
89	0.0178 4240	0.0186 8490	0.0195 4756	0.0204 2984	0.0222 5102
90	0.0177 2684	0.0185 7146	0.0194 3641	0.0203 2113	0.0221 4760
91	0.0176 1403	0.0184 6076	0.0193 2799	0.0202 1516	0.0220 4690
92	0.0175 0387	0.0983 5271	0.0192 2222	0.0201 1182	0.0219 4882
93	0.0173 9629	0.0182 4724	0.0191 1902	0.0200 1104	0.0218 5327
94	0.0172 9119	0.0181 4425	0.0190 1829	0.0199 1273	0.0217 6017
95	0.0171 8851	0.0180 4366	0.0189 1997	0.0198 1681	0.0216 6944
96	0.0170 8816	0.0179 4540	0.0188 2397	0.0197 2321	0.0215 8101
97	0.0169 9007	0.0178 4941	0.0187 3022	0.0196 3186	0.0214 9480
98	0.0168 9418	0.0177 5560	0.0186 3866	0.0195 4268	0.0214 1074
99	0.0168 0041	0.0176 6391	0.0185 4921	0.0194 5560	0.0213 2876
100	0.0167 0870	0.0175 7428	0.0184 6181	0.0193 7057	0.0212 4880

TABLE VII. The Annuity Whose Present Value at Compound Interest is 1

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n}|i}}$$

n	2%	$2\frac{1}{4}$%	$2\frac{1}{2}$%	$2\frac{3}{4}$%	3%
1	1.0200 0000	1.0225 0000	1.0250 0000	1.0275 0000	1.0300 0000
2	0.5150 4950	0.5169 3758	0.5188 2716	0.5207 1825	0.5226 1084
3	0.3467 5467	0.3484 4458	0.3501 3717	0.3518 3243	0.3535 3036
4	0.2626 2375	0.2642 1893	0.2658 1788	0.2674 2059	0.2690 2705
5	0.2121 5839	0.2137 0021	0.2152 4686	0.2167 9832	0.2183 5457
6	0.1785 2581	0.1800 3496	0.1815 4997	0.1830 7083	0.1845 9750
7	0.1545 1196	0.1560 0025	0.1574 9543	0.1589 9747	0.1605 0635
8	0.1365 0980	0.1379 8462	0.1394 6735	0.1409 5795	0.1424 5639
9	0.1225 1544	0.1239 8170	0.1254 5689	0.1269 4095	0.1284 3386
10	0.1113 2653	0.1127 8768	0.1142 5876	0.1157 3972	0.1172 3051
11	0.1021 7794	0.1036 3649	0.1051 0596	0.1065 8629	0.1080 7745
12	0.0945 5960	0.0960 1740	0.0974 8713	0.0989 6871	0.1004 6209
13	0.0881 1835	0.0895 7686	0.0910 4827	0.0925 3252	0.0940 2954
14	0.0826 0197	0.0840 6230	0.0855 3653	0.0870 2457	0.0885 2634
15	0.0778 2547	0.0792 8852	0.0807 6646	0.0822 5917	0.0837 6658
16	0.0736 5013	0.0751 1663	0.0765 9899	0.0780 9710	0.0796 1085
17	0.0699 6984	0.0714 4039	0.0729 2777	0.0744 3186	0.0759 5253
18	0.0667 0210	0.0681 7720	0.0696 7008	0.0711 8063	0.0727 0870
19	0.0637 8177	0.0652 6182	0.0667 6062	0.0682 7802	0.0698 1388
20	0.0611 5672	0.0626 4207	0.0641 4713	0.0656 7173	0.0672 1571
21	0.0587 8477	0.0602 7572	0.0617 8733	0.0633 1941	0.0648 7178
22	0.0566 3140	0.0581 2821	0.0596 4661	0.0611 8640	0.0627 4739
23	0.0546 6810	0.0561 7097	0.0576 9638	0.0592 4410	0.0608 1390
24	0.0528 7110	0.0543 8023	0.0559 1282	0.0574 6863	0.0590 4742
25	0.0512 2044	0.0527 3599	0.0542 7592	0.0558 3997	0.0574 2787
26	0.0496 9923	0.0512 2134	0.0527 6875	0.0543 4116	0.0559 3829
27	0.0482 9309	0.0498 2188	0.0513 7687	0.0529 5776	0.0545 6421
28	0.0469 8967	0.0485 2525	0.0500 8793	0.0516 7738	0.0532 9323
29	0.0457 7836	0.0473 2081	0.0488 9127	0.0504 8935	0.0521 1467
30	0.0446 4992	0.0461 9934	0.0477 7764	0.0493 8442	0.0510 1926
31	0.0435 9635	0.0451 5280	0.0467 3000	0.0483 5453	0.0499 9893
32	0.0426 1061	0.0441 7415	0.0457 6831	0.0473 9263	0.0490 4662
33	0.0416 8653	0.0432 5722	0.0448 5938	0.0464 9253	0.0481 5612
34	0.0408 1867	0.0423 9655	0.0440 0675	0.0456 4875	0.0473 2196
35	0.0400 0221	0.0415 8731	0.0432 0558	0.0448 5645	0.0465 3929
36	0.0392 3285	0.0408 2522	0.0424 5158	0.0441 1132	0.0458 0379
37	0.0385 0678	0.0401 0643	0.0417 4090	0.0434 0953	0.0451 1162
38	0.0378 5207	0.0394 2753	0.0410 7012	0.0427 4764	0.0444 5934
39	0.0371 7114	0.0387 8543	0.0404 3615	0.0421 2256	0.0438 4385
40	0.0365 5575	0.0381 7738	0.0398 3623	0.0415 3151	0.0432 6238
41	0.0359 7188	0.0376 0087	0.0392 6786	0.0409 7200	0.0427 1241
42	0.0354 1729	0.0370 5364	0.0387 2876	0.0404 4175	0.0421 9167
43	0.0348 8993	0.0365 3364	0.0382 1688	0.0399 3871	0.0416 9811
44	0.0343 8794	0.0360 3901	0.0377 3037	0.0394 6100	0.0412 2985
45	0.0339 0962	0.0355 6805	0.0372 6752	0.0390 0693	0.0407 8518
46	0.0334 5342	0.0351 1921	0.0368 2676	0.0385 7493	0.0403 6254
47	0.0330 1792	0.0346 9107	0.0364 0669	0.0381 6358	0.0399 6051
48	0.0326 0184	0.0342 8233	0.0360 0599	0.0377 7158	0.0395 7777
49	0.0322 0396	0.0338 9179	0.0356 2348	0.0373 9773	0.0392 1314
50	0.0318 2321	0.0335 1836	0.0352 5806	0.0370 4092	0.0388 6550

TABLE VII. The Annuity Whose Present Value at Compound Interest is 1

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n}|i}}$$

n	2%	$2\frac{1}{4}$%	$2\frac{1}{2}$%	$2\frac{3}{4}$%	3%
51	0.0314 5856	0.0331 6102	0.0349 0870	0.0367 0014	0.0385 3382
52	0.0311 0909	0.0328 1884	0.0345 7446	0.0363 7444	0.0382 1718
53	0.0307 7392	0.0324 9094	0.0342 5449	0.0360 6297	0.0379 1471
54	0.0304 5226	0.0321 7654	0.0339 4799	0.0357 6491	0.0376 2558
55	0.0301 4337	0.0318 7489	0.0336 5419	0.0354 7953	0.0373 4907
56	0.0298 4656	0.0315 8530	0.0333 7243	0.0352 0612	0.0370 8447
57	0.0295 6120	0.0313 0712	0.0331 0204	0.0349 4404	0.0368 3114
58	0.0292 8667	0.0310 3977	0.0328 4244	0.0346 9270	0.0365 8848
59	0.0290 2243	0.0307 8268	0.0325 9307	0.0344 5153	0.0363 5593
60	0.0287 6797	0.0305 3533	0.0323 5340	0.0342 2002	0.0361 3296
61	0.0285 2278	0.0302 9724	0.0321 2294	0.0339 9767	0.0359 1908
62	0.0282 8643	0.0300 6795	0.0319 0126	0.0337 8402	0.0357 1385
63	0.0280 5848	0.0298 4704	0.0316 8790	0.0335 7866	0.0355 1682
64	0.0278 3855	0.0296 3411	0.0314 8249	0.0333 8118	0.0353 2760
65	0.0276 2624	0.0294 2878	0.0312 8463	0.0331 9120	0.0351 4581
66	0.0274 2122	0.0292 3070	0.0310 9398	0.0330 0837	0.0349 7110
67	0.0272 2316	0.0290 3955	0.0309 1021	0.0328 3236	0.0348 0313
68	0.0270 3173	0.0288 5500	0.0307 3300	0.0326 6285	0.0346 4159
69	0.0268 4665	0.0286 7677	0.0305 6206	0.0324 9955	0.0344 8618
70	0.0266 6765	0.0285 0458	0.0303 9712	0.0323 4218	0.0343 3663
71	0.0264 9446	0.0283 3816	0.0302 3790	0.0321 9048	0.0341 9266
72	0.0263 2683	0.0281 7728	0.0300 8417	0.0320 4420	0.0340 5404
73	0.0261 6454	0.0280 2169	0.0299 3568	0.0319 0311	0.0339 2053
74	0.0260 0736	0.0278 7118	0.0297 9222	0.0317 6698	0.0337 9191
75	0.0258 5508	0.0277 2554	0.0296 5358	0.0316 3560	0.0336 6796
76	0.0257 0751	0.0275 8457	0.0295 1956	0.0315 0878	0.0335 4849
77	0.0255 6447	0.0274 4808	0.0293 8997	0.0313 8633	0.0334 3331
78	0.0254 2576	0.0273 1589	0.0292 6463	0.0312 6806	0.0333 2224
79	0.0252 9123	0.0271 8784	0.0291 4338	0.0311 5382	0.0332 1510
80	0.0251 6071	0.0270 6376	0.0290 2605	0.0310 4342	0.0331 1175
81	0.0250 3405	0.0269 4350	0.0289 1248	0.0309 3674	0.0330 1201
82	0.0249 1110	0.0268 2692	0.0288 0254	0.0308 3361	0.0329 1576
83	0.0247 9173	0.0267 1387	0.0286 9608	0.0307 3389	0.0328 2284
84	0.0246 7581	0.0266 0423	0.0285 9298	0.0306 3747	0.0327 3313
85	0.0245 6321	0.0264 9787	0.0284 9310	0.0305 4420	0.0326 4650
86	0.0244 5381	0.0263 9467	0.0283 9633	0.0304 5397	0.0325 6284
87	0.0243 4750	0.0262 9452	0.0283 0255	0.0303 6667	0.0324 8202
88	0.0242 4416	0.0261 9730	0.0282 1165	0.0302 8219	0.0324 0393
89	0.0241 4370	0.0261 0291	0.0281 2353	0.0302 0041	0.0323 2848
90	0.0240 4602	0.0260 1126	0.0280 3809	0.0301 2125	0.0322 5556
91	0.0239 5101	0.0259 2224	0.0279 5523	0.0300 4460	0.0321 8508
92	0.0238 5859	0.0258 3577	0.0278 7486	0.0299 7038	0.0321 1694
93	0.0237 6868	0.0257 5176	0.0277 9690	0.0298 9850	0.0320 5107
94	0.0236 8118	0.0256 7012	0.0277 2126	0.0298 2887	0.0319 8737
95	0.0235 9602	0.0255 9078	0.0276 4786	0.0297 6141	0.0319 2577
96	0.0235 1313	0.0255 1366	0.0275 7662	0.0296 9605	0.0318 6619
97	0.0234 3242	0.0254 3868	0.0275 0747	0.0296 3272	0.0318 0856
98	0.0233 5383	0.0253 6578	0.0274 4034	0.0295 7134	0.0317 5281
99	0.0232 7729	0.0252 9489	0.0273 7517	0.0295 1185	0.0316 9886
100	0.0232 0274	0.0252 2594	0.0273 1188	0.0294 5418	0.0316 4667

TABLE VII. The Annuity Whose Present Value at Compound Interest is 1

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n}|i}}$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$
1	1.0350 0000	1.0400 0000	1.0450 0000	1.0500 0000	1.0550 0000
2	0.5264 0049	0.5301 9608	0.5339 9756	0.5378 0488	0.5416 1800
3	0.3569 3418	0.3603 4854	0.3637 7336	0.3672 0856	0.3706 5407
4	0.2722 5114	0.2754 9005	0.2787 4365	0.2820 1183	0.2852 9449
5	0.2214 8137	0.2246 2711	0.2277 9164	0.2309 7480	0.2341 7644
6	0.1876 6821	0.1907 6190	0.1938 7839	0.1970 1747	0.2001 7895
7	0.1635 4449	0.1666 0961	0.1697 0147	0.1728 1982	0.1759 6442
8	0.1454 7665	0.1485 2783	0.1516 0965	0.1547 2181	0.1578 6401
9	0.1314 4601	0.1344 9299	0.1375 7447	0.1406 9008	0.1438 3946
10	0.1202 4137	0.1232 9094	0.1263 7882	0.1295 0458	0.1326 6777
11	0.1110 9197	0.1141 4904	0.1172 4818	0.1203 8889	0.1235 7065
12	0.1034 8395	0.1065 5217	0.1096 6619	0.1128 2541	0.1160 2923
13	0.0970 6157	0.1001 4373	0.1032 7535	0.1064 5577	0.1096 8426
14	0.0915 7073	0.0946 6897	0.0978 2032	0.1010 2397	0.1042 7912
15	0.0868 2507	0.0899 4110	0.0931 1381	0.0963 4229	0.0996 2560
16	0.0826 8483	0.0858 2000	0.0890 1537	0.0922 6991	0.0955 8254
17	0.0790 4313	0.0821 9852	0.0854 1758	0.0886 9914	0.0920 4197
18	0.0758 1684	0.0789 9333	0.0822 3690	0.0855 4622	0.0889 1992
19	0.0729 4033	0.0761 3862	0.0794 0734	0.0827 4501	0.0861 5006
20	0.0703 6108	0.0735 8175	0.0768 7614	0.0802 4259	0.0836 7933
21	0.0680 3659	0.0712 8011	0.0746 0057	0.0779 9611	0.0814 6478
22	0.0659 3207	0.0691 9881	0.0725 4565	0.0759 7051	0.0794 7123
23	0.0640 1880	0.0673 0906	0.0706 8249	0.0741 3682	0.0776 6965
24	0.0622 7283	0.0655 8683	0.0689 8703	0.0724 7090	0.0760 3580
25	0.0606 7404	0.0640 1196	0.0674 3903	0.0709 5246	0.0745 4935
26	0.0592 0540	0.0625 6738	0.0660 2137	0.0695 6432	0.0731 9307
27	0.0578 5241	0.0612 3854	0.0647 1946	0.0682 9186	0.0719 5228
28	0.0566 0265	0.0600 1298	0.0635 2081	0.0671 2253	0.0708 1440
29	0.0554 4538	0.0588 7993	0.0624 1461	0.0660 4551	0.0697 6857
30	0.0543 7133	0.0578 3010	0.0613 9154	0.0650 5144	0.0688 0539
31	0.0533 7240	0.0568 5535	0.0604 4345	0.0641 3212	0.0679 1665
32	0.0524 4150	0.0559 4859	0.0595 6320	0.0632 8042	0.0670 9519
33	0.0515 7242	0.0551 0357	0.0587 4453	0.0624 9004	0.0663 3469
34	0.0507 5966	0.0543 1477	0.0579 8191	0.0617 5545	0.0656 2958
35	0.0499 9835	0.0535 7732	0.0572 7045	0.0610 7171	0.0649 7493
36	0.0492 8416	0.0528 8688	0.0566 0578	0.0604 3446	0.0643 6635
37	0.0486 1325	0.0522 3957	0.0559 8402	0.0598 3979	0.0637 9993
38	0.0479 8214	0.0516 3192	0.0554 0169	0.0592 8423	0.0632 7217
39	0.0473 8775	0.0510 6083	0.0548 5567	0.0587 6462	0.0627 7991
40	0.0468 2728	0.0505 2349	0.0543 4315	0.0582 7816	0.0623 2034
41	0.0462 9822	0.0500 1738	0.0538 6158	0.0578 2229	0.0618 9090
42	0.0457 9828	0.0495 4020	0.0534 0868	0.0573 9471	0.0614 8927
43	0.0453 2539	0.0490 8989	0.0529 8235	0.0569 9333	0.0611 1337
44	0.0448 7768	0.0486 6454	0.0525 8071	0.0566 1625	0.0607 6128
45	0.0444 5343	0.0482 6246	0.0522 0202	0.0562 6173	0.0604 3127
46	0.0440 5108	0.0478 8205	0.0518 4471	0.0559 2820	0.0601 2175
47	0.0436 6919	0.0475 2189	0.0515 0734	0.0556 1421	0.0598 3129
48	0.0433 0646	0.0471 8065	0.0511 8858	0.0553 1843	0.0595 5854
49	0.0429 6167	0.0468 5712	0.0508 8722	0.0550 3965	0.0593 0230
50	0.0426 3371	0.0465 5020	0.0506 0215	0.0547 7674	0.0590 6145

TABLE VII. The Annuity Whose Present Value at Compound Interest is 1

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n}|i}}$$

n	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$
51	0.0423 2156	0.0462 5885	0.0503 3232	0.0545 2867	0.0588 3495
52	0.0420 2429	0.0459 8212	0.0500 7679	0.0542 9450	0.0586 2186
53	0.0417 4100	0.0457 1915	0.0498 3469	0.0540 7334	0.0584 2130
54	0.0414 7090	0.0454 6910	0.0496 0519	0.0538 6438	0.0582 3245
55	0.0412 1323	0.0452 3124	0.0493 8754	0.0536 6686	0.0580 5458
56	0.0409 6730	0.0450 0487	0.0491 8105	0.0534 8010	0.0578 8698
57	0.0407 3245	0.0447 8932	0.0489 8506	0.0533 0343	0.0577 2900
58	0.0405 0810	0.0445 8401	0.0487 9897	0.0531 3626	0.0575 8006
59	0.0402 9366	0.0443 8836	0.0486 2221	0.0529 7802	0.0574 3959
60	0.0400 8862	0.0442 0185	0.0484 5426	0.0528 2818	0.0573 0707
61	0.0398 9249	0.0440 2398	0.0482 9462	0.0526 8627	0.0571 8202
62	0.0397 0480	0.0438 5430	0.0481 4284	0.0525 5183	0.0570 6400
63	0.0395 2513	0.0436 9237	0.0479 9848	0.0524 2442	0.0569 5258
64	0.0393 5308	0.0435 3780	0.0478 6115	0.0523 0365	0.0568 4737
65	0.0391 8826	0.0433 9019	0.0477 3047	0.0521 8915	0.0567 4800
66	0.0390 3031	0.0432 4921	0.0476 0608	0.0520 8057	0.0566 5413
67	0.0388 7892	0.0431 1451	0.0474 8765	0.0519 7757	0.0565 6544
68	0.0387 3375	0.0429 8578	0.0473 7487	0.0518 7986	0.0564 8163
69	0.0385 9453	0.0428 6272	0.0472 6745	0.0517 8715	0.0564 0242
70	0.0384 6095	0.0427 4506	0.0471 6511	0.0516 9915	0.0563 2754
71	0.0383 3277	0.0426 3253	0.0470 6759	0.0516 1563	0.0562 5675
72	0.0382 0973	0.0425 2489	0.0469 7465	0.0515 3633	0.0561 8982
73	0.0380 9160	0.0424 2190	0.0468 8606	0.0514 6103	0.0561 2652
74	0.0379 7816	0.0423 2334	0.0468 0159	0.0513 8953	0.0560 6665
75	0.0378 6919	0.0422 2900	0.0467 2104	0.0513 2161	0.0560 1002
76	0.0377 6450	0.0421 3869	0.0466 4422	0.0512 5709	0.0559 5645
77	0.0376 6390	0.0420 5221	0.0465 7094	0.0511 9580	0.0559 0577
78	0.0375 6721	0.0419 6939	0.0465 0104	0.0511 3756	0.0558 5781
79	0.0374 7426	0.0418 9007	0.0464 3434	0.0510 8222	0.0558 1243
80	0.0373 8489	0.0418 1408	0.0463 7069	0.0510 2962	0.0557 6948
81	0.0372 9894	0.0417 4127	0.0463 0995	0.0509 7963	0.0557 2884
82	0.0372 1628	0.0416 7150	0.0462 5197	0.0509 3211	0.0556 9036
83	0.0371 3676	0.0416 0463	0.0461 9663	0.0508 8694	0.0556 5395
84	0.0370 6025	0.0415 4054	0.0461 4379	0.0508 4399	0.0556 1947
85	0.0369 8662	0.0414 7909	0.0460 9334	0.0508 0316	0.0555 8683
86	0.0369 1576	0.0414 2018	0.0460 4516	0.0507 6433	0.0555 5593
87	0.0368 4756	0.0413 6370	0.0459 9915	0.0507 2740	0.0555 2667
88	0.0367 8190	0.0413 0953	0.0459 5522	0.0506 9228	0.0554 9896
89	0.0367 1868	0.0412 5758	0.0459 1325	0.0506 5888	0.0554 7273
90	0.0366 5781	0.0412 0775	0.0458 7316	0.0506 2711	0.0554 4788
91	0.0365 9919	0.0411 5995	0.0458 3486	0.0505 9689	0.0554 2435
92	0.0365 4273	0.0411 1410	0.0457 9827	0.0505 6815	0.0554 0207
93	0.0364 8834	0.0410 7010	0.0457 6331	0.0505 4080	0.0553 8096
94	0.0364 3594	0.0410 2789	0.0457 2991	0.0505 1478	0.0553 6097
95	0.0363 8546	0.0409 8738	0.0456 9799	0.0504 9003	0.0553 4204
96	0.0363 3682	0.0409 4850	0.0456 6749	0.0504 6648	0.0553 2410
97	0.0362 8995	0.0409 1119	0.0456 3834	0.0504 4407	0.0553 0711
98	0.0362 4478	0.0408 7538	0.0456 1048	0.0504 2274	0.0552 9101
99	0.0362 0124	0.0408 4100	0.0455 8385	0.0504 0245	0.0552 7577
100	0.0361 5927	0.0408 0800	0.0455 5839	0.0503 8314	0.0552 6132

TABLE VII. The Annuity Whose Present Value at Compound Interest is 1

$$\frac{1}{a_{\overline{n|}i}} = \frac{i}{1-(1+i)^{-n}} = i + \frac{1}{s_{\overline{n|}i}}$$

n	6%	$6\frac{1}{2}$%	7%	$7\frac{1}{2}$%	8%
1	1.0600 0000	1.0650 0000	1.0700 0000	1.0750 0000	1.0800 0000
2	0.5454 3689	0.5492 6150	0.5530 9179	0.5569 2771	0.5607 6923
3	0.3741 0981	0.3775 7570	0.3810 5166	0.3845 3763	0.3880 3351
4	0.2885 9149	0.2919 0274	0.2952 2812	0.2985 6751	0.3019 2080
5	0.2373 9640	0.2406 3454	0.2438 9069	0.2471 6472	0.2504 5645
6	0.2033 6263	0.2065 6831	0.2097 9580	0.2130 4489	0.2163 1539
7	0.1791 3502	0.1823 3137	0.1855 5322	0.1888 0032	0.1920 7240
8	0.1610 3594	0.1642 3730	0.1674 6776	0.1707 2702	0.1740 1476
9	0.1470 2224	0.1502 3803	0.1534 8647	0.1567 6716	0.1600 7971
10	0.1358 6796	0.1391 0469	0.1423 7750	0.1456 8593	0.1490 2949
11	0.1267 9294	0.1300 5521	0.1333 5690	0.1366 9747	0.1400 7634
12	0.1192 7703	0.1225 6817	0.1259 0199	0.1292 7783	0.1326 9502
13	0.1129 6011	0.1162 8256	0.1196 5085	0.1230 6420	0.1265 2181
14	0.1075 8491	0.1109 4048	0.1143 4494	0.1177 9737	0.1212 9685
15	0.1029 6276	0.1063 5278	0.1097 9462	0.1132 8724	0.1168 2954
16	0.0989 5214	0.1023 7757	0.1058 5765	0.1093 9116	0.1129 7687
17	0.0954 4480	0.0989 0633	0.1024 2519	0.1060 0003	0.1096 2943
18	0.0923 5654	0.0958 5461	0.0994 1260	0.1030 2896	0.1067 0210
19	0.0896 2086	0.0931 5575	0.0967 5301	0.1004 1090	0.1041 2763
20	0.0871 8456	0.0907 5640	0.0943 9293	0.0980 9219	0.1018 5221
21	0.0850 0455	0.0886 1333	0.0922 8900	0.0960 2937	0.0998 3225
22	0.0830 4557	0.0866 9120	0.0904 0577	0.0941 8687	0.0980 3207
23	0.0812 7848	0.0849 6078	0.0887 1393	0.0925 3528	0.0964 2217
24	0.0796 7900	0.0833 9770	0.0871 8902	0.0910 5008	0.0949 7796
25	0.0782 2672	0.0819 8148	0.0858 1052	0.0897 1067	0.0936 7878
26	0.0769 0435	0.0806 9480	0.0845 6103	0.0884 9961	0.0925 0713
27	0.0756 9717	0.0795 2288	0.0834 2573	0.0874 0204	0.0914 4809
28	0.0745 9255	0.0784 5305	0.0823 9193	0.0864 0520	0.0904 8891
29	0.0735 7961	0.0774 7440	0.0814 4865	0.0854 9811	0.0896 1854
30	0.0726 4891	0.0765 7744	0.0805 8640	0.0846 7124	0.0888 2743
31	0.0717 9222	0.0757 5393	0.0797 9691	0.0839 1628	0.0881 0728
32	0.0710 0234	0.0749 9665	0.0790 7292	0.0832 2599	0.0874 5081
33	0.0702 7293	0.0742 9924	0.0784 0807	0.0825 9397	0.0868 5163
34	0.0695 9843	0.0736 5610	0.0777 9674	0.0820 1461	0.0863 0411
35	0.0689 7386	0.0730 6226	0.0772 3396	0.0814 8291	0.0858 0326
36	0.0683 9483	0.0725 1332	0.0767 1531	0.0809 9447	0.0853 4467
37	0.0678 5743	0.0720 0534	0.0762 3685	0.0805 4533	0.0849 2440
38	0.0673 5812	0.0715 3480	0.0757 9505	0.0801 3197	0.0845 3894
39	0.0668 9377	0.0710 9854	0.0753 8676	0.0797 5124	0.0841 8513
40	0.0664 6154	0.0706 9373	0.0750 0914	0.0794 0031	0.0838 6016
41	0.0660 5886	0.0703 1779	0.0746 5962	0.0790 7663	0.0835 6149
42	0.0656 8342	0.0699 6842	0.0743 3591	0.0787 7789	0.0832 8684
43	0.0653 3312	0.0696 4352	0.0740 3590	0.0785 0201	0.0830 3414
44	0.0650 0606	0.0693 4119	0.0737 5769	0.0782 4710	0.0828 0152
45	0.0647 0050	0.0690 5968	0.0734 9957	0.0780 1146	0.0825 8728
46	0.0644 1485	0.0687 9743	0.0732 5996	0.0777 9353	0.0823 8991
47	0.0641 4768	0.0685 5300	0.0730 3744	0.0775 9190	0.0822 0799
48	0.0638 9766	0.0683 2506	0.0728 3070	0.0774 0527	0.0820 4027
49	0.0636 6356	0.0681 1240	0.0726 3853	0.0772 3247	0.0818 8557
50	0.0634 4429	0.0679 1393	0.0724 5985	0.0770 7241	0.0817 4286

TABLE VIII. The Amount of 1 at Compound Interest for Fractional Periods

$$(1+i)^{\frac{1}{p}}$$

p	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{8}\%$	1%
2	1.0016 6528	1.0020 8117	1.0024 9688	1.0043 6547	1.0049 8756
3	1.0011 0988	1.0013 8696	1.0016 6390	1.0029 0820	1.0033 2228
4	1.0008 3229	1.0010 4004	1.0012 4766	1.0021 8036	1.0024 9068
6	1.0005 5479	1.0006 9324	1.0008 3160	1.0014 5304	1.0016 5977
12	1.0002 7735	1.0003 4656	1.0004 1571	1.0007 2626	1.0008 2954
13	1.0002 5602	1.0003 1990	1.0003 8373	1.0006 7037	1.0007 6570
26	1.0001 2800	1.0001 5994	1.0001 9185	1.0003 3513	1.0003 8276
52	1.0000 6400	1.0000 7996	1.0000 9592	1.0001 6755	1.0001 9137

p	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
2	1.0056 0927	1.0062 3059	1.0068 5153	1.0074 7208	1.0087 1205
3	1.0037 3602	1.0041 4943	1.0045 6249	1.0049 7521	1.0057 9963
4	1.0028 0081	1.0031 1046	1.0034 1992	1.0037 2909	1.0043 4658
6	1.0018 6627	1.0020 7257	1.0022 7865	1.0024 8452	1.0028 9562
12	1.0009 3270	1.0010 3575	1.0011 3868	1.0012 4149	1.0014 4677
13	1.0008 6092	1.0009 5604	1.0010 5104	1.0011 4594	1.0013 3540
26	1.0004 3037	1.0004 7790	1.0005 2538	1.0005 7280	1.0006 6748
52	1.0002 1516	1.0002 3892	1.0002 6266	1.0002 8636	1.0003 3368

p	2%	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%
2	1.0099 5050	1.0111 8742	1.0124 2284	1.0136 5675	1.0148 8916
3	1.0066 2271	1.0074 4444	1.0082 6484	1.0090 8390	1.0099 0163
4	1.0049 6293	1.0055 7815	1.0061 9225	1.0068 0522	1.0074 1707
6	1.0033 0589	1.0037 1532	1.0041 2392	1.0045 3168	1.0049 3862
12	1.0016 5158	1.0018 5594	1.0020 5984	1.0022 6328	1.0024 6627
13	1.0015 2444	1.0017 1305	1.0019 0124	1.0020 8900	1.0022 7634
26	1.0007 6193	1.0008 5616	1.0009 5017	1.0010 4396	1.0011 3752
52	1.0003 8089	1.0004 2799	1.0004 7497	1.0005 2184	1.0005 6860

p	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$
2	1.0173 4950	1.0198 0390	1.0222 5242	1.0246 9508	1.0271 3193
3	1.0115 3314	1.0131 5941	1.0147 8046	1.0163 9636	1.0180 0713
4	1.0086 3745	1.0098 5341	1.0110 6499	1.0122 7224	1.0134 7518
6	1.0057 5004	1.0065 5820	1.0073 6312	1.0081 6485	1.0089 6340
12	1.0028 7090	1.0032 7374	1.0036 7481	1.0040 7412	1.0044 7170
13	1.0026 4977	1.0030 2153	1.0033 9165	1.0037 6014	1.0041 2701
26	1.0013 2401	1.0015 0963	1.0016 9439	1.0018 7831	1.0020 6138
52	1.0006 6179	1.0007 5453	1.0008 4684	1.0009 3871	1.0010 3016

p	6%	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%
2	1.0295 6302	1.0319 8837	1.0344 0804	1.0368 2207	1.0392 3048
3	1.0196 1282	1.0212 1347	1.0228 0912	1.0243 9981	1.0259 8557
4	1.0146 7385	1.0158 6828	1.0170 5853	1.0182 4460	1.0194 2655
6	1.0097 5880	1.0105 5107	1.0113 4026	1.0121 2638	1.0129 0946
12	1.0048 6755	1.0052 6169	1.0056 5415	1.0060 4492	1.0064 3403
13	1.0044 9228	1.0048 5597	1.0052 1808	1.0055 7863	1.0059 3764
26	1.0022 4363	1.0024 2504	1.0026 0564	1.0027 8544	1.0029 6443
52	1.0011 2118	1.0012 1179	1.0013 0197	1.0013 9175	1.0014 8112

TABLE IX. Nominal Rate of Interest j with Frequency of Conversion p Corresponding to Effective Rate of Interest i

$$j_{(p)} = p\left[(1+i)^{\frac{1}{p}} - 1\right]$$

p	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{8}\%$	1%
2	.0033 3056	.0041 6234	.0049 9377	.0087 3094	.0099 7512
3	.0033 2964	.0041 6089	.0049 9169	.0087 2460	.0099 6685
4	.0033 2917	.0041 6017	.0049 9065	.0087 2143	.0099 6272
6	.0033 2871	.0041 5945	.0049 8962	.0087 1827	.0099 5859
12	.0033 2825	.0041 5873	.0049 8858	.0087 1510	.0099 5446
13	.0033 2822	.0041 5868	.0049 8850	.0087 1486	.0099 5414
26	.0033 2800	.0041 5834	.0049 8802	.0087 1340	.0099 5224
52	.0033 2790	.0041 5818	.0048 8778	.0087 1267	.0099 5128
p	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
2	.0112 1854	.0124 6118	.0137 0360	.0149 4417	.0174 2410
3	.0112 0807	.0124 4828	.0136 8746	.0149 2562	.0173 9890
4	.0112 0285	.0124 4183	.0136 7966	.0149 1636	.0173 8631
6	.0111 9763	.0124 3539	.0136 7188	.0149 0710	.0173 7374
12	.0111 9241	.0124 2895	.0136 6410	.0148 9785	.0173 6119
13	.0111 9200	.0124 2846	.0136 6350	.0148 9714	.0173 6022
26	.0111 8960	.0124 2549	.0136 5991	.0148 9288	.0173 5443
52	.0111 8839	.0124 2400	.0136 5812	.0148 9074	.0173 5153
p	2%	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%
2	.0199 0099	.0223 7484	.0248 4567	.0273 1349	.0297 7831
3	.0198 6813	.0223 3333	.0247 9451	.0272 5170	.0297 0490
4	.0198 5173	.0223 1261	.0247 6899	.0272 2087	.0296 6829
6	.0198 3534	.0222 9192	.0247 4349	.0271 9009	.0296 3173
12	.0198 1898	.0222 7125	.0247 1804	.0271 5936	.0295 9524
13	.0198 1772	.0222 6966	.0247 1608	.0271 5699	.0295 9243
26	.0198 1017	.0222 6013	.0247 0434	.0271 4283	.0295 7561
52	.0198 0640	.0222 5537	.0246 9848	.0271 3575	.0295 6721
p	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$
2	.0346 9899	.0396 0781	.0445 0483	.0493 9015	.0542 6386
3	.0345 9943	.0394 7821	.0443 4138	.0491 8907	.0540 2139
4	.0345 4978	.0394 1363	.0442 5996	.0490 8894	.0539 0070
6	.0345 0024	.0393 4918	.0441 7874	.0489 8908	.0537 8036
12	.0344 5078	.0392 8488	.0440 9771	.0488 8949	.0536 6039
13	.0344 4698	.0392 7994	.0440 9149	.0488 8184	.0536 5117
26	.0344 2420	.0392 5031	.0440 5417	.0488 3597	.0535 9593
52	.0344 1281	.0392 3551	.0440 3552	.0488 1306	.0535 6834
p	6%	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%
2	.0591 2603	.0639 7674	.0688 1609	.0736 4414	.0784 6097
3	.0588 3847	.0636 4042	.0684 2737	.0731 9942	.0779 5670
4	.0586 9538	.0634 7314	.0682 3410	.0729 7840	.0777 0619
6	.0585 5277	.0633 0644	.0680 4156	.0727 5827	.0774 5674
12	.0584 1061	.0631 4033	.0678 4974	.0725 3903	.0772 0836
13	.0583 9969	.0631 2758	.0678 3502	.0725 2220	.0771 8930
26	.0583 3425	.0630 5113	.0677 4676	.0724 2134	.0770 7506
52	.0583 0157	.0630 1295	.0677 0268	.0723 7098	.0770 1802

TABLE X. Amount at End of Year at Compound Interest of p Instalments Each of $\frac{1}{p}$ Deposited at End of Each pth Part of a Year

$$\frac{i}{j_{(p)}} = s_{\overline{1}|i}^{(p)} = \frac{1}{p s_{\overline{\frac{1}{p}}|i}} = \frac{i}{p[(1+i)^{\frac{1}{p}} - 1]}$$

p	$\frac{1}{3}\%$	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{8}\%$	1%
2	1.0008 3264	1.0010 4058	1.0012 4844	1.0021 8274	1.0024 9378
3	1.0011 1029	1.0013 8761	1.0016 6482	1.0029 1102	1.0033 2596
4	1.0012 4913	1.0015 6115	1.0018 7305	1.0032 7529	1.0037 4223
6	1.0013 8799	1.0017 3471	1.0020 8131	1.0036 3967	1.0041 5861
12	1.0015 2686	1.0019 0829	1.0022 8960	1.0040 0411	1.0045 7510
13	1.0015 3754	1.0019 2164	1.0023 0563	1.0040 3215	1.0046 0714
26	1.0016 0164	1.0020 0176	1.0024 2182	1.0042 0039	1.0047 9941
52	1.0016 3369	1.0020 4183	1.0024 4985	1.0042 8452	1.0048 9556
p	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{3}{8}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$
2	1.0028 0463	1.0031 1529	1.0034 2576	1.0037 3604	1.0043 6176
3	1.0037 4068	1.0041 5516	1.0045 6942	1.0049 8346	1.0058 1084
4	1.0042 0892	1.0046 7537	1.0051 4158	1.0056 0755	1.0065 3878
6	1.0046 7730	1.0051 9575	1.0057 1395	1.0062 3191	1.0072 6707
12	1.0051 4583	1.0057 1632	1.0062 8654	1.0068 5652	1.0079 9571
13	1.0051 8188	1.0057 5637	1.0063 3060	1.0069 0458	1.0080 5177
26	1.0053 9818	1.0059 9669	1.0065 9495	1.0071 9296	1.0083 8820
52	1.0055 0634	1.0061 1687	1.0067 2715	1.0073 3717	1.0085 5644
p	2%	$2\frac{1}{4}\%$	$2\frac{1}{2}\%$	$2\frac{3}{4}\%$	3%
2	1.0049 7525	1.0055 9371	1.0062 1142	1.0068 2837	1.0074 4458
3	1.0066 3733	1.0074 6292	1.0082 8761	1.0091 1141	1.0099 3431
4	1.0074 6856	1.0083 9839	1.0093 2677	1.0102 5422	1.0111 8072
6	1.0083 0125	1.0093 3444	1.0103 6665	1.0113 9789	1.0124 2816
12	1.0091 3389	1.0102 7107	1.0114 0725	1.0125 4243	1.0136 7662
13	1.0091 9796	1.0103 4314	1.0114 8732	1.0126 3051	1.0137 7270
26	1.0095 8243	1.0107 7565	1.0119 6786	1.0131 5908	1.0143 4929
52	1.0097 7470	1.0109 9195	1.0122 0819	1.0134 2343	1.0146 3757
p	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	$5\frac{1}{2}\%$
2	1.0086 7475	1.0099 0195	1.0111 2621	1.0123 4754	1.0135 6596
3	1.0115 7748	1.0132 1713	1.0148 5328	1.0164 8597	1.0181 1522
4	1.0130 3094	1.0148 7744	1.0167 2026	1.0185 5942	1.0203 9495
6	1.0144 8578	1.0165 3957	1.0185 8953	1.0206 3570	1.0226 7810
12	1.0159 4203	1.0182 0351	1.0204 6109	1.0227 1479	1.0249 6465
13	1.0160 5410	1.0183 3158	1.0206 0515	1.0228 7484	1.0251 4068
26	1.0167 2674	1.0191 0023	1.0214 6980	1.0238 3548	1.0261 9729
52	1.0170 6316	1.0194 8470	1.0219 6231	1.0243 1602	1.0267 2586
p	6%	$6\frac{1}{2}\%$	7%	$7\frac{1}{2}\%$	8%
2	1.0147 8151	1.0159 9419	1.0172 0402	1.0184 1103	1.0196 1524
3	1.0197 4104	1.0213 6348	1.0229 8254	1.0245 9826	1.0262 1065
4	1.0222 2688	1.0240 5523	1.0258 8002	1.0277 0129	1.0295 1904
6	1.0247 1676	1.0267 5172	1.0287 8298	1.0308 1059	1.0328 3456
12	1.0272 1070	1.0294 5294	1.0316 9143	1.0339 2617	1.0361 5721
13	1.0274 0270	1.0296 6093	1.0319 1538	1.0341 6609	1.0364 1309
26	1.0285 5526	1.0309 0941	1.0332 5978	1.0356 0640	1.0379 4927
52	1.0291 3186	1.0315 3404	1.0339 3242	1.0363 2705	1.0387 1794

TABLE XI. American Experience Table of Mortality

Age	Number living	Number of deaths	Yearly probability of dying	Yearly probability of living	Complete expectation of life
x	l_x	d_x	q_x	p_x	$\mathring{e}_x$
10	100,000	749	0.007 490	0.992 510	48.72
11	99,251	746	0.007 516	0.992 484	48.08
12	98,505	743	0.007 543	0.992 457	47.45
13	97,762	740	0.007 569	0.992 431	46.80
14	97,022	737	0.007 596	0.992 404	46.16
15	96,285	735	0.007 634	0.992 366	45.50
16	95,550	732	0.007 661	0.992 339	44.85
17	94,818	729	0.007 688	0.992 312	44.19
18	94,089	727	0.007 727	0.992 273	43.53
19	93,362	725	0.007 765	0.992 235	42.87
20	92,637	723	0.007 805	0.992 195	42.20
21	91,914	722	0.007 855	0.992 145	41.53
22	91,192	721	0.007 906	0.992 094	40.85
23	90,471	720	0.007 958	0.992 042	40.17
24	89,751	719	0.008 011	0.991 989	39.49
25	89,032	718	0.008 065	0.991 935	38.81
26	88,314	718	0.008 130	0.991 870	38.12
27	87,596	718	0.008 197	0.991 803	37.43
28	86,878	718	0.008 264	0.991 736	36.73
29	86,160	719	0.008 345	0.991 655	36.03
30	85,441	720	0.008 427	0.991 573	35.33
31	84,721	721	0.008 510	0.991 490	34.63
32	84,000	723	0.008 607	0.991 393	33.92
33	83,277	726	0.008 718	0.991 282	33.21
34	82,551	729	0.008 831	0.991 169	32.50
35	81,822	732	0.008 946	0.991 054	31.78
36	81,090	737	0.009 089	0.990 911	31.07
37	80,353	742	0.009 234	0.990 776	30.35
38	79,611	749	0.009 408	0.990 592	29.62
39	78,862	756	0.009 586	0.990 414	28 90
40	78,106	765	0.009 794	0.990 206	28.18
41	77,341	774	0.010 008	0.989 992	27.45
42	76,567	785	0.010 252	0.989 748	26.72
43	75,782	797	0.010 517	0.989 483	26.00
44	74,985	812	0.010 829	0.989 171	25.27
45	74,173	828	0.011 163	0.988 837	24.54
46	73,345	848	0.011 562	0.988 438	23.81
47	72,497	870	0.012 000	0.988 000	23.08
48	71,627	896	0.012 509	0.987 491	22.36
49	70,731	927	0.013 106	0.986 894	21.63
50	69,804	962	0.013 781	0.986 219	20.91
51	68,842	1,001	0.014 541	0.985 459	20.20
52	67,841	1,044	0.015 389	0.984 611	19.49
53	66,797	1,091	0.016 333	0.983 667	18.79
54	65,706	1,143	0.017 396	0.982 604	18.09
55	64,563	1,199	0.018 571	0.981 429	17.40
56	63,364	1,260	0.019 885	0.980 115	16 72
57	62,104	1,325	0.021 335	0.978 665	16.05
58	60,779	1,394	0.022 936	0.977 064	15.39
59	59,385	1,468	0.024 720	0.975 280	14.74
60	57,917	1,546	0.026 693	0.973 307	14.10
61	56,371	1,628	0.028 880	0.971 120	13.47
62	54,743	1,713	0.031 292	0.968 708	12.86
63	53,030	1,800	0.033 943	0.966 057	12.26
64	51,230	1.889	0.036 873	0.963 127	11.67
65	49,341	1,980	0.040 129	0.959 871	11.10
66	47,361	2,070	0.043 707	0.956 293	10.54
67	45,291	2,158	0.047 647	0.952 353	10.00
68	43,133	2,243	0.052 002	0.947 998	9.47
69	40,890	2,321	0.056 762	0.943 238	8.97
70	38,569	2,391	0.061 993	0.938 007	8.48
71	36,178	2,448	0.067 665	0.932 335	8.00
72	33,730	2,487	0.073 733	0.926 267	7.55
73	31,243	2,505	0.080 178	0.919 822	7.11
74	28,738	2,501	0.087 028	0.912 972	6.68
75	26,237	2,476	0.094 371	0.905 629	6.27
76	23,761	2,431	0.102 311	0.897 689	5.88
77	21,330	2,369	0.111 064	0.888 936	5.49
78	18,961	2,291	0.120 827	0.879 173	5.11
79	16,670	2,196	0.131 734	0.868 266	4.74
80	14,474	2,091	0.144 466	0.855 534	4.39
81	12,383	1,964	0.158 605	0.841 395	4.05
82	10,419	1,816	0.174 297	0.825 703	3.71
83	8,603	1,648	0.191 561	0.808 439	3.39
84	6,955	1,470	0.211 359	0.788 641	3.08
85	5,485	1,292	0.235 552	0.764 448	2.77
86	4,193	1,114	0.265 681	0.734 319	2.47
87	3,079	933	0.303 020	0.696 980	2.18
88	2,146	744	0.346 692	0.653 308	1.91
89	1,402	555	0.395 863	0.604 137	1.66
90	847	385	0.454 545	0.545 455	1.42
91	462	246	0.532 468	0.467 534	1.19
92	216	137	0.634 259	0.365 741	.98
93	79	58	0.734 177	0.265 823	.80
94	21	18	0.857 143	0.142 857	.64
95	3	3	1.000 000	0.000 000	.50

TABLE XII. Commutation Columns, American Experience Table, $3\frac{1}{2}\%$

Age x	D_x	N_x	C_x	M_x	$a_x = \frac{N_x}{D_x} = 1 + a_x$	$A_x = \frac{M_x}{D_x}$
10	70 891.9	1 575 535.3	513.02	17 612.91	22.2245	0.24845
11	67 981.5	1 504 643.4	493.69	17 099.89	22.1331	0.25154
12	65 189.0	1 436 661.9	475.08	16 606.20	22.0384	0.25474
13	62 509.4	1 371 472.9	457.16	16 131.12	21.9403	0.25806
14	59 938.4	1 308 963.5	439.91	15 673.96	21.8385	0.26151
15	54 471.6	1 249 025.0	423.88	15 234.05	21.7329	0.26508
16	55 104.2	1 191 553.4	407.87	14 810.17	21.6236	0.26877
17	52 832.9	1 136 449.2	392.47	14 402.30	21.5102	0.27261
18	50 653.9	1 083 616.2	378.15	14 009.83	21.3926	0.27659
19	48 562.8	1 032 962.4	364.36	13 631.68	21.2707	0.28071
20	46 556.2	984 399.6	351.07	13 267.32	21.1443	0.28497
21	44 630.8	937 843.4	338.73	12 916.25	21.0134	0.28940
22	42 782.8	893 212.6	326.82	12 577.53	20.8779	0.29399
23	41 009.2	850 429.9	315.33	12 250.71	20.7375	0.29873
24	39 307.1	809 420.6	304.24	11 935.38	20.5922	0.30365
25	37 673.6	770 113.6	293.55	11 631.14	20.4417	0.30873
26	36 106.1	732 439.9	283.62	11 337.59	20.2858	0.31401
27	34 601.5	696 333.8	274.03	11 053.97	20.1244	0.31947
28	33 157.4	661 732.4	264.76	10 779.94	19.9573	0.32512
29	31 771.3	628 575.0	256.16	10 515.18	19.7843	0.33097
30	30 440.8	596 803.6	247.85	10 259.02	19.6054	0.33702
31	29 163.5	566 362.9	239.797	10 011.17	19.4202	0.34328
32	27 937.5	537 199.3	232.331	9 771.375	19.2286	0.34976
33	26 760.5	509 261.8	225.406	9 539.044	19.0304	0.35646
34	25 630.1	482 501.3	218.683	9 313.638	18.8256	0.36339
35	24 544.7	456 871.2	212.157	9 094.955	18.6138	0.37055
36	23 502.5	432 326.5	206.383	8 882.798	18.3949	0.37795
37	22 501.4	408 824.0	200.757	8 676.415	18.1688	0.38560
38	21 539.7	386 322.6	195.798	8 475.658	17.9354	0.39349
39	20 615.5	364 782.9	190.945	8 279.860	17.6946	0.40163
40	19 727.4	344 167.4	186.684	8 088.915	17.4461	0.41003
41	18 873.6	324 440.0	182.493	7 902.231	17.1901	0.41869
42	18 052.9	305 566.3	178.828	7 719.738	16.9262	0.42762
43	17 263.6	287 513.4	175.421	7 540.910	16.6543	0.43681
44	16 504.4	270 249.8	172.680	7 365.489	16.3744	0.44628
45	15 773.6	253 745.5	170.127	7 192.809	16.0867	0.45600
46	15 070.0	237 971.9	168.345	7 022.682	15.7911	0.46600
47	14 392.1	222 901.9	166.872	6 854.337	15.4878	0.47626
48	13 738.5	208 509.8	166.047	6 687.466	15.1770	0.48677
49	13 107.9	194 771.3	165.983	6 521.419	14.8591	0.49752
50	12 498.6	181 663.4	166.424	6 355.436	14.5346	0.50849
51	11 909.6	169 164.7	167.316	6 189.012	14.2041	0.51967
52	11 339.5	157 255.2	168.601	6 021.696	13.8679	0.53104

TABLE XII. Commutation Columns, American Experience Table, $3\frac{1}{2}\%$

Age x	D_x	N_x	C_x	M_x	$\mathbf{a}_x = \frac{N_x}{D_x} = 1 + a_x$	$A_x = \frac{M_x}{D_x}$
53	10 787.4	145 915.7	170.234	5 853.095	13.5264	0.54258
54	10 252.4	135 128.2	172.317	5 682.861	13.1801	0.55430
55	9 733.40	124 875.8	174.646	5 510.544	12.8296	0.56615
56	9 229.60	115 142.4	177.325	5 335.898	12.4753	0.57813
57	8 740.17	105 912.8	180.168	5 158.573	12.1179	0.59022
58	8 264.44	97 172.64	183.139	4 978.405	11.7579	0.60239
59	7 801.83	88 908.20	186.340	4 795.266	11.3958	0.61463
60	7 351.65	81 106.38	189.604	4 608.926	11.0324	0.62692
61	6 913.44	73 754.73	192.909	4 419.322	10.6683	0.63924
62	6 486.75	66 841.28	196 117	4 226.413	10.3043	0.65155
63	6 071.27	60 354.54	199.109	4 030.296	9.9410	0.66383
64	5 666.85	54 283.27	201.887	3 831.187	9.5791	0.67607
65	5 273.33	48 616.41	204.457	3 629.300	9.2193	0.68824
66	4 890.55	43 343.08	206.522	3 424.843	8.8626	0.70030
67	4 518.65	38 452.53	208.022	3 218.321	8.5097	0.71223
68	4 157.82	33 933.88	208.903	3 010.299	8.1615	0.72401
69	3 808.32	29 776.06	208.858	2 801.396	7 8187	0.73560
70	3 470.67	25 967.74	207.881	2 592.538	7.4820	0.74698
71	3 145.43	22 497.07	205.639	2 384.657	7.1523	0.75813
72	2 833.42	19 351.64	201.851	2 179.018	6.8298	0.76904
73	2 535.75	16 518.22	196.436	1 977.167	6.5141	0.77972
74	2 253.57	13 982.47	189.491	1 780.731	6.2046	0.79018
75	1 987.87	11 728.90	181.253	1 591.240	5.9002	0.80048
76	1 739.39	9 741.028	171.940	1 409.988	5.6002	0.81062
77	1 508.63	8 001.633	161.889	1 238.047	5.3039	0.82064
78	1 295.73	6 492.999	151.264 6	1 076.158	5.0111	0.83054
79	1 100.65	5 197.271	140.089 1	924.893 7	4.7220	0.84032
80	923.338	4 096.624	128.880 1	784.804 6	4.4368	0.84997
81	763.234	3 173.286	116.958 8	655.924 5	4.1577	0.85940
82	620.465	2 410.052	104.488	538.965 7	3.8843	0.86865
83	494.995	1 789.587	91.615 2	434.477 6	3.6154	0.87774
84	386.641	1 294.592	78.956 5	342.862 4	3.3483	0.88677
85	294.610	907.951 3	67.049 0	263.905 9	3.0819	0.89578
86	217.598	613.341 7	55.856 6	196.856 9	2.8187	0.90468
87	154.383	395.743 8	45.199 2	141.000 3	2 5634	0.91332
88	103.963	241.360 9	34.824 26	95.801 07	2.3216	0.92149
89	65.623 1	137.397 8	25.099 29	60.976 82	2.0937	0.92920
90	38.304 7	71.774 70	16.822 44	35.877 52	1.8738	0.93664
91	20.186 9	33.470 01	10.385 393	19.055 09	1.6580	0.94393
92	9.118 89	13.283 09	5.588 150	8.669 695	1.4567	0.95074
93	3.222 36	4.164 21	2.285 484	3.081 545	1.2923	0.95630
94	0.827 611	0.941 84	0.685 393	.795 762	1.1380	0.96152
95	0.114 232	0.114 23	0.110 369	.110 369	1.0000	0.96618

TABLE XIII. Valuation Columns, American Experience Table, $3\frac{1}{2}\%$

$$u_x = \frac{D_x}{D_{x+1}}, \quad k_x = \frac{C_x}{D_{x+1}}$$

Age x	u_x	k_x	Age x	u_x	k_x
10	1.042 811	0.007 546	53	1.052 185	0.016 604
11	1.042 838	0.007 573	54	1.053 323	0.017 704
12	1.042 866	0.007 600	55	1.054 585	0.018 922
13	1.042 894	0.007 627	56	1.055 999	0.020 289
14	1.042 922	0.007 654	57	1.057 563	0.021 800
15	1.042 962	0.007 692	58	1.059 296	0.023 474
16	1.042 990	0.007 720	59	1.061 234	0.025 347
17	1.043 019	0.007 748	60	1.063 385	0.027 425
18	1.043 059	0.007 787	61	1.065 780	0.029 739
19	1.043 100	0.007 826	62	1.068 433	0.032 303
20	1.043 141	0.007 866	63	1.071 365	0.035 136
21	1.043 195	0.007 917	64	1.074 625	0.038 285
22	1.043 248	0.007 969	65	1.078 270	0.041 807
23	1.043 303	0.008 022	66	1.082 304	0.045 704
24	1.043 358	0.008 076	67	1.086 782	0.050 031
25	1.043 415	0.008 130	68	1.091 774	0.054 855
26	1.043 484	0.008 197	69	1.097 284	0.060 178
27	1.043 554	0.008 264	70	1.103 403	0.066 090
28	1.043 625	0.008 333	71	1.110 117	0.072 576
29	1.043 710	0.008 415	72	1.117 388	0.079 602
30	1.043 796	0.008 498	73	1.125 218	0.087 167
31	1.043 884	0.008 583	74	1.133 660	0.095 323
32	1.043 986	0.008 682	75	1.142 852	0.104 204
33	1.044 102	0.008 795	76	1.152 960	0.113 971
34	1.044 221	0.008 910	77	1.164 314	0.124 941
35	1.044 343	0.009 027	78	1.177 243	0.137 433
36	1.044 493	0.009 172	79	1.192 031	0.151 720
37	1.044 647	0.009 320	80	1.209 771	0.168 861
38	1.044 830	0.009 498	81	1.230 099	0.188 502
39	1.045 018	0.009 679	82	1.253 477	0.211 089
40	1.045 238	0.009 891	83	1.280 245	0.236 952
41	1.045 463	0.010 109	84	1.312 384	0.268 004
42	1.045 721	0.010 359	85	1.353 917	0.308 133
43	1.046 001	0.010 629	86	1.409 469	0.361 806
44	1.046 331	0.010 947	87	1.484 979	0.434 762
45	1.046 684	0.011 289	88	1.584 244	0.530 671
46	1.047 106	0.011 697	89	1.713 188	0.655 254
47	1.047 571	0.012 146	90	1.897 500	0.833 333
48	1.048 111	0.012 668	91	2.213 750	1.138 889
49	1.048 745	0.013 280	92	2.829 873	1.734 177
50	1.049 463	0.013 974	93	3.893 571	2.761 905
51	1.050 272	0.014 755	94	7.245 000	6.000 000
52	1.051 177	0.015 629	95		

TABLE XIV. Commutation Columns, Two Lives, Equal Ages. Hunter's Makehamized American Experience Table of Mortality, $3\frac{1}{2}\%$

Age x	l_x	μ_x	D_{xx}	N_{xx}	M_{xx}
10	100 081	0.007 68	71 006.79	1 351 270.60	25 312.24
11	99 315	0.007 67	67 559.44	1 280 263.81	24 266.06
12	98 553	0.007 70	64 276.96	1 212 704.37	23 268.17
13	97 796	0.007 72	61 152.96	1 148 427.41	22 317.76
14	97 044	0.007 73	58 179.86	1 087 274.45	21 412.61
15	96 296	0.007 75	55 349.23	1 029 094.59	20 549.39
16	95 552	0.007 76	52 654.32	973 745.36	19 726.17
17	94 812	0.007 78	50 088.83	921 091.04	18 941.24
18	94 076	0.007 81	47 646.53	871 002.21	18 192.74
19	93 343	0.007 83	45 320.75	823 355.68	17 478.18
20	92 614	0.007 86	43 106.86	778 034.93	16 796.86
21	91 888	0.007 88	40 998.73	734 928.07	16 146.43
22	91 165	0.007 92	38 991.41	693 929.34	15 525.52
23	90 444	0.007 95	37 079.33	654 937.93	14 931.97
24	89 726	0.007 99	35 258.85	617 858.60	14 365.36
25	89 010	0.008 04	33 525.00	582 599.75	13 823.82
26	88 295	0.008 09	31 873.04	549 074.75	13 305.54
27	87 581	0.008 14	30 299.13	517 201.71	12 809.45
28	86 868	0.008 21	28 799.80	486 902.58	12 334.72
29	86 156	0.008 27	27 371.63	458 102.78	11 880.44
30	85 442	0.008 35	26 009.48	430 731.15	11 443.89
31	84 728	0.008 43	24 711.70	404 721.67	11 025.65
32	84 013	0.008 53	23 474.81	380 009.97	10 624.41
33	83 295	0.008 63	22 294.96	356 535.16	10 238.37
34	82 575	0.008 75	21 170.22	334 240.20	9 867.57
35	81 850	0.008 88	20 096.73	313 069.98	9 509.97
36	81 121	0.009 02	19 072.80	292 973.25	9 165.63
37	80 387	0.009 18	18 095.86	273 900.45	8 833.66
38	79 646	0.009 35	17 163.06	255 804.59	8 512.79
39	78 896	0.009 55	16 271.86	238 641.53	8 201.98
40	78 138	0.009 77	15 420.90	222 369.67	7 901.27
41	77 369	0.010 01	14 607.60	206 948.77	7 609.44
42	76 589	0.010 28	13 830.50	192 341.17	7 326.31
43	75 794	0.010 58	13 086.84	178 510.67	7 050.34
44	74 985	0.010 91	12 375.79	165 423.83	6 781.83
45	74 158	0.011 28	11 694.99	153 048.04	6 519.53
46	73 311	0.011 69	11 042.88	141 353.05	6 262.90
47	72 443	0.012 15	10 418.30	130 310.17	6 011.75
48	71 551	0.012 65	9 819.595	119 891.87	5 765.35
49	70 631	0.013 21	9 245.132	110 072.28	5 522.95
50	69 683	0.013 84	8 694.313	100 827.14	5 284.76
51	68 702	0.014 53	8 165.446	92 132.83	5 049.60
52	67 685	0.015 31	7 657.482	83 967.384	4 818.06
53	66 628	0.016 17	7 169.279	76 309.902	4 588.80
54	65 529	0.017 12	6 700.222	69 140.623	4 362.18

TABLE XIV. Commutation Columns, Two Lives, Equal Ages. Hunter's Makehamized American Experience Table of Mortality, $3\frac{1}{2}\%$

Age x	l_x	μ_x	D_{xx}	N_{xx}	M_{xx}
55	64 383	0.018 18	6249.176	62 440.401	4 137.71
56	63 187	0.019 36	5815.616	56 191.225	3 915.47
57	61 936	0.020 66	5 398.652	50 375.609	3 695.17
58	60 626	0.022 12	4 997.777	44 976.957	3 476.86
59	59 253	0.023 73	4 612.539	39 979.180	3 260.63
60	57 812	0.025 53	4 242.422	35 366.641	3 046.49
61	56 300	0.027 52	3 887.371	31 124.219	2 834.90
62	54 711	0.029 74	3 546.903	27 236.848	2 625.89
63	53 044	0.032 20	3 221.281	23 689.945	2 420.21
64	51 294	0.034 94	2 910.389	20 468.664	2 218.25
65	49 459	0.037 98	2 614.369	17 558.275	2 020.65
66	47 536	0.041 36	2 333.359	14 943.906	1 828.05
67	45 526	0.045 12	2 067.829	12 610.547	1 641.42
68	43 429	0.049 29	1 818.085	10 542.718	1 461.60
69	41 246	0.053 93	1 584.460	8 724.633	1 289.45
70	38 982	0.059 08	1 367.424	7 140.173	1 125.99
71	36 642	0.064 81	1 167.328	5 772.749	972.13
72	34 235	0.071 17	984.544 9	4 605.420 8	828.82
73	31 773	0.078 24	819.349 4	3 620.875 9	696.96
74	29 269	0.086 10	671.786 8	2 801.526 5	577.05
75	26 739	0.094 83	541.707 3	2 129.739 7	469.69
76	24 204	0.104 53	428.853 7	1 588.032 4	375.151
77	21 687	0.115 31	332.652 1	1 159.178 7	293.452
78	19 211	0.127 29	252.206 2	826.526 6	224.255
79	16 805	0.140 60	186.462 7	574.320 4	167.041
80	14 495	0.155 40	134.032 3	387.857 7	120.916
81	12 309	0.171 83	93.385 61	253.825 4	84.801 6
82	10 273	0.190 10	62.846 87	160.439 8	57.420 6
83	8 410	0.210 40	40.695 54	97.592 91	37.394 6
84	6 739	0.232 95	25.246 62	56.897 37	23.322 6
85	5 274	0.258 01	14.940 03	31.650 75	13.869 7
86	4 019	0.285 86	8.382 428	16.710 72	7.817 29
87	2 974	0.316 81	4.434 808	8.328 292	4.153 09
88	2 130	0.351 20	2.197 902	3.893 484	2.066 19
89	1 471	0.389 41	1.012 828	1.695 582	.955 492
90	976	0.431 87	.430 792 9	.682 754	.407 702
91	619	0.479 05	.167 420 9	.251 962	.158 902
92	374	0.531 49	.059 051 5	.084 541 2	.056 191 8
93	214	0.589 75	.018 679 9	.025 489 7	.017 817 8
94	115	0.654 49	.005 211 9	.006 809 84	.004 981 78
95	58	0.726 43	.001 280 9	.001 597 94	.001 226 58
96	27	0.806 37	.000 268 20	.000 317 038	.000 257 479
97	11	0.895 21	.000 043 011	.000 048 837 9	.000 041 359 2
98	4	0.993 92	.000 005 495 0	.000 005 826 82	.000 005 293 17
99	1		.000 000 331 82	.000 000 331 82	.000 000 320 60

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